

## BGWM as second constituent of complex matrix model

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## BGWM as second constituent of complex matrix model

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ABSTRACT: In [1] we explained that partition functions of various matrix models can be constructed from that of the cubic Kontsevich model, which, therefore, becomes a basic elementary building block in "M-theory" of matrix models [2]. However, the less topical complex matrix model appeared to be an exception: its decomposition involved not only the Kontsevich  $\tau$ -function but also another constituent, which we now identify as the Brezin-Gross-Witten (BGW) partition function. The BGW  $\tau$ -function can be represented either as a generating function of all unitary-matrix integrals or as a Kontsevich-Penner model with potential  $1/X$  (instead of  $X^3$  in the cubic Kontsevich model).

KEYWORDS: Matrix Models, M-Theory, Integrable Hierarchies, Topological Strings

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## Introduction

Matrix models [3, 4] play a very special role in modern theoretical physics. They appear regularly and prove useful in analysis of various simplified models of concrete physical phenomena, but their real significance is that they somehow capture and reflect the very basic properties of string theory — and can serve to represent the universal classes of quantum field theory models. From the very beginning matrix models were introduced to describe some very general features (eigenvalue repulsion) of statistical distributions [5]. It was, perhaps, the first recognition of the role of group theory – the underlying theory behind matrix models – in explaining the fundamental properties of quantum/statistical behavior. Much later this led to discovery that integrability is the basic property of all functional integrals, considered as functionals on the moduli space of theories [6], and matrix models played a central role [7] in the formulation of the fundamental relation

$$\text{partition function} = \tau - \text{function} \tag{1}$$

between the two central concepts of modern theory, already with a variety of applications in different fields, from gauge theories [8] to Hurwitz theory [9] and with still many more to come. An immediate implication of (1) is that quantitative approach – a possibility to *calculate* something – in string theory (= a theory of *families* of quantum mechanical models) requires extension of the standard set of special functions to a broader set of  $\tau$ -functions [10, 11] – a far-going generalization of both hypergeometric and elliptic families.

A highly non-trivial step here was introduction of "infinite-genus"  $\tau$ -functions, satisfying the string equations [12] and Virasoro/ $W$ -constraints [13]–[21], and it was once again inspired by the study of matrix models. Unfortunately, even the simplest of these  $\tau$ -functions, associated with Hermitian [3] and Kontsevich [22–25] matrix models, are not yet systematically studied/tabulated and still can not be included into the special-functions textbooks — see [26] for the first attempts in this direction. It is very important to realize that the world of such  $\tau$ -functions is cognizable, and is, perhaps, actually finitely-generated: many (all?) matrix-model  $\tau$ -functions are actually expressed by group-theoretical methods through a few basic ones. This decomposition results from description of genus expansion of matrix-model partition functions in terms of auxiliary "spectral" Riemann surfaces [26–28], and explicitly relates them to representation theory of Krichever-Novikov type deformation [29] of Kac-Moody algebras. One of spectacular byproducts of this development is the possibility to build a "string-field-theory-like" diagram technique [28] for the model of entire string theory, provided by "M-theory of matrix models" [2]. As shown in [1], the main basic block (constituent) of the matrix model partition functions in this approach is the ordinary Kontsevich  $\tau$ -function  $Z_K$  of [22].

However, already in [1] a first counter-example was found to this (over?)-optimistic conjecture: partition function of the complex matrix model [30, 31] is made not only from  $Z_K$ , but also from some other ingredient, denoted  $\tilde{Z}_K$  in s.8 of [1]. The purpose of the present paper is to identify this  $\tilde{Z}_K$  with a very important and well-known partition function: that of Brezin-Gross-Witten model (BGWM) [20, 32, 33]:

$$\tilde{Z}_K = Z_{\text{BGW}} \tag{2}$$

By definition, BGWM describes correlators of unitary matrices with a non-linear Haar measure. Unitary correlators play a crucially important role in description of gluons in lattice gauge-theory models [34], however, unitary matrix models are more complicated than Hermitian ones, they are in intermediate position between eigenvalue and non-eigenvalue models and remain under-investigated, see [35]–[38] for some crucial references. A modern matrix-model-theory approach to BGWM and its embedding into the set of generalized Kontsevich models (GKM) [23] was outlined in [20], but has not been developed any further since then. Hopefully reappearance of this model in the context of matrix-model M-theory will help to attract new attention to this unjustly-abandoned subject.

We begin in section I from reminding the definition and the main properties of the four partition functions which participate in our story:  $Z_H(t)$ ,  $Z_C(t)$ ,  $Z_K(\tau)$  and  $Z_{\text{BGW}}(\tau)$ . They were originally introduced as matrix integrals over Hermitian ( $Z_H$  and  $Z_K$ ), complex ( $Z_C$ ) and unitary ( $Z_{\text{BGW}}$ ) matrices, with the time-variables identified either with the coupling constants ( $t_k$  in  $Z_H$  and  $Z_C$ ) or with the Miwa transform of the background field ( $\tau_k$  in  $Z_K$  and  $Z_{\text{BGW}}$ ).

As functions of their parameters — the time-variables  $t$  or  $\tau$  — these integrals satisfy Ward identities (or Picard-Fuchs equations) [16], which have the form of the Virasoro constraints. Namely,

$$\frac{\partial Z_H}{\partial t_k} = \hat{L}_{k-2} Z_H, \quad \frac{\partial Z_C}{\partial t_k} = \hat{L}_{k-1} Z_C \tag{3}$$

with  $k \geq 1$ , with  $\frac{\partial Z_H}{\partial t_0} = N/gZ_H$ ,  $\frac{\partial Z_C}{\partial t_0} = N/gZ_C$ , where  $N$  is the size of the matrix in the original integral representation and with "discrete-Virasoro" operators [15]

$$\hat{L}_m = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{m+k}} + g^2 \sum_{\substack{a,b=0 \\ a+b=m}}^m \frac{\partial^2}{\partial t_a \partial t_b}, \quad m \geq -1 \quad (4)$$

and

$$\frac{\partial Z_K}{\partial \tau_{k-1}} = \hat{L}_{k-2} Z_K, \quad \frac{\partial Z_{BGW}}{\partial \tau_{k-1}} = \hat{L}_{k-1} Z_{BGW} \quad (5)$$

with  $k \geq 1$  and with "continuous-Virasoro" operators [13, 14]

$$\hat{\mathcal{L}}_m = \sum_{k=1}^{\infty} \left(k + \frac{1}{2}\right) \tau_k \frac{\partial}{\partial \tau_{m+k}} + g^2 \sum_{\substack{a,b=0 \\ a+b=m-1}}^{m-1} \frac{\partial^2}{\partial \tau_a \partial \tau_b} + \frac{\tau_0^2}{16g^2} \delta_{m,-1} + \frac{1}{16} \delta_{m,0}, \quad m \geq -1 \quad (6)$$

Now one can switch to  $D$ -module approach and define the four partition functions as solutions to the four systems of linear differential equations (3) and (5), and original integral formulas are just integral representations for the solutions.

Not surprisingly, such representations are not unique, and one can instead represent the same solutions in a very different integral form: of Kontsevich-Penner integrals over  $n \times n$  Hermitian matrices with a peculiar Penner term  $N \text{tr} \log \phi$  in the action. This puts all the four models in the unifying context of GKM theory [23]. Direct relation between the two integral representations is provided by a version of Faddeev-Popov trick from [21].

All four matrix integrals can be expressed in the form of determinants of the other matrices, which have ordinary single integrals as their elements. These determinant representations are very important, because they are typical for the *tau-functions* of integrable hierarchies [24, 40] – the generalized characters of Lie algebras [11, 39]. In other words, partition functions of the matrix models are always the tau-functions [7]. Moreover, this is a general property of all partition functions — the generating functions of all correlation functions in any quantum theory, — this is a consequence of the freedom to change integration variables (fields) in the functional integral [6]. For tau-functions the whole sets of Virasoro constraints are actually fixed by their lowest components  $\hat{L}_{-1}$  or  $\hat{L}_0$ , which therefore has its own name: the *string equation* [12].

Integrability means that partition function satisfies a bilinear Hirota equation [10] of the form

$$\oint Z_N \left(t_k + \frac{1}{kz^k}\right) Z_{N'} \left(t'_k - \frac{1}{kz^k}\right) z^{N'-N} e^{\sum_k (t'_k - t_k) z^k} dz \quad (7)$$

This equation has its origin in decomposition rule  $R \times R' = \sum_I R_I$  for representations of Lie algebras and this is an equation for the *characters* of the algebra [11, 39]. For loop algebras the characters can be rather non-trivial, they can be actually labeled by some auxiliary (spectral) Riemann surfaces or, better, by the points of an infinite-dimensional Grassmannian (the *universal moduli space* of [41]) — what means that the spectral surface can actually have an infinite genus (and this is typically the case for the matrix-model partition functions).

For irreducible representations of finite-dimensional simple Lie algebras, the characters are given by two determinant Weyl formulas: in the case of  $SL(N)$  and representation labeled by partition  $\vec{m} : m_1 \geq m_2 \geq \dots \geq m_N \geq 0$  (the Young diagram)

$$\chi_{\vec{m}}(t) = \det P_{i+m_j-1}(t), \tag{8}$$

where  $P_i(t)$  are the Shur polynomials,  $\exp(\sum_k t_k x^k) = \sum_m x^m P_m(t)$ , and, after the Miwa transform  $t_k = \frac{1}{k} \sum_i \lambda_i^{-k}$ ,

$$\chi_{\vec{m}}(t) = \frac{\det_{ij} \lambda_i^{j+m_j}}{\Delta(\lambda)}, \tag{9}$$

In fact, the two Weyl formulas are mirrored in two possible representations of matrix models partition function: as we shall demonstrate, the Hermitian and complex matrix models, besides the standard determinant representation of type (8), have a Kontsevich-Penner representation of type (9).

Virasoro constraints can be used as recursion relations to provide the logarithm of the partition functions  $g^2 \log Z$  in the form of the formal series in non-negative powers of  $t$ -variables and  $g^2$ . Such formal series are unambiguously defined by the systems (3) and (5), since  $k \geq 1$  in all the four cases. The generating functions ("multiresolvents")  $\rho^{(p|q)}(z)$ , defined as

$$\rho^{(p|q)}(z_1, \dots, z_q) = \hat{\nabla}(z_1) \dots \hat{\nabla}(z_q) \log Z|_{t=0} = \sum_{p=0}^{\infty} g^{2p-2} \rho^{(p|q)}(z_1, \dots, z_q) \tag{10}$$

possess an important property: they are poly-differentials on auxiliary *spectral* Riemann surfaces (complex curves) [26, 28], which for the four matrix models in question are all double-coverings of the Riemann sphere with only two ramification points (and thus Riemann spheres themselves). The spectral curve representation arises only for the special choice of generating functions: they should be *resolvents*, i.e. no  $k$ -dependent coefficients are allowed in (10). There are other interesting choices of coefficients, when alternative generating functions possess other interesting properties, see for example [42].

In section II we proceed to decomposition formulas. Virasoro constraints can be also considered as quadratic differentials on the spectral surfaces, expanded near particular points. It turns out that "discrete" operators (4) arise in expansion near non-singular points, while "continuous" operators (6) — in those near ramification points of degree 2. This means that a globally-defined Virasoro quadratic differential can be decomposed in both bases and this idea finally leads to decomposition formulas [1].

The basic one is

$$Z_H = \hat{U}_{KK}(Z_K \otimes Z_K), \tag{11}$$

it expresses  $Z_H(t)$  for Hermitian matrix model through  $Z_C(\tau)$  for Kontsevich model. It is explained in full detail in section 7. Ingredients of the construction are: explicit parametrization of the spectral curve and of the singular differentials in the vicinities of particular points on it, explicit formula for the global  $\hat{U}(1)$  current and the Virasoro differential, its projection onto "canonical" quadratic differentials in the vicinities of the

particular points and Bogoliubov transform of the time-variables with the help of the conjugation  $\hat{U}$ -operator, and, finally, projection from generic Laurent series for the Virasoro operator to a Taylor series, provided by peculiar projection operator  $\mathcal{P}$ , which picks up a triangular subalgebra from entire Virasoro (Krichever-Novikov) algebra. Generators of this triangular subalgebra can be imposed as constraints on partition function and form a consistent and resolvable set of constraints.

From the point of view of  $D$ -module approach the difference between  $Z_H$  and  $Z_C$  in (3) looks minor: both are annihilated by *the same* discrete-Virasoro operators  $\hat{L}_n$ , only  $n \geq -1$  for  $Z_H$ , but  $n \geq 0$  for  $Z_C$ . The second difference is that the shift of time-variables, which generates the l.h.s. in (3), is also different:  $t_2$  is shifted in the case of  $Z_H$ , but  $t_1$  is shifted in the case of  $Z_C$  — this is important to explain why both sets of equations (3) have unambiguous formal-series solution, despite the set of constraints contains one less equation in the case of  $Z_C$ . However, this second difference is not essential for comparison of (3) and (5). Indeed, the relation between  $Z_K$  and  $\tilde{Z}_K = Z_{\text{BGW}}$  in (5) is exactly the same: both partition functions are annihilated by *the same* continuous-Virasoro operators  $\mathcal{L}_n$ , but  $n \geq -1$  for  $Z_K$ , while  $n \geq 0$  for  $Z_{\text{BGW}}$ . This time shifted are  $\tau_1$  in  $Z_K$  and  $\tau_0$  in  $Z_{\text{BGW}}$ , what guarantees that the formal-series solutions are unambiguously defined in both cases. All this implies that in the spectral-surface formalism the difference between  $Z_H$  and  $Z_C$  is concentrated in the choice of projection operator  $\mathcal{P}$  at the last stage. Since projection operator can be realized as a contour integral, its modification can actually be shifted from  $Z_H$  (where it transformed  $Z_H$  into  $Z_C$ ) to one of the two  $Z_K$  at the r.h.s. of (11) and convert it into  $\tilde{Z}_K = Z_{\text{BGW}}$ . In other words, instead of (11) we obtain

$$Z_C = \hat{U}_{K\tilde{K}}(Z_K \otimes \tilde{Z}_K) \quad \text{i.e.} \quad Z_C = \hat{U}_{K\tilde{K}}(Z_K \otimes Z_{\text{BGW}}) \quad (12)$$

This is the main result of the present paper and it is discussed in full detail in section 8. This formula was supported in [1] by explicit comparison of the first terms of expansions of partition functions at both sides of (12).

Some concluding remarks are given in section 8.

## Part I

# The four matrix models

### 1 Hermitian matrix model $Z_H(t)$

#### 1.1 Original integral representation

Partition function of Hermitian matrix model [3] is defined by the integral over hermitian  $N \times N$  matrix

$$Z_H(t) = \int_{N \times N} \exp \left( -\frac{1}{2g} \text{Tr} H^2 + \frac{1}{g} \sum_{k=0}^{\infty} t_k \text{Tr} H^k \right) dH \quad (1.1)$$

where the measure  $dH = \prod_{i,j=1}^N dH_{ij}$ . This is nothing but the generating function of all  $GL(N)$ -invariant Gaussian correlators of Hermitian matrix  $H$ .



### 1.2 Eigenvalue representation

The Hermitian matrix  $\Phi$  can be diagonalized by a unitary transformation,  $H = UDU^\dagger$ , where  $D = \text{diag}(H_1, \dots, H_N)$  is the diagonal matrix made from the eigenvalues of  $H$ . The norm of  $H$  decomposes as

$$\text{Tr} (\delta H)^2 = \text{Tr} (\delta D)^2 + \text{Tr} \left( [U^\dagger \delta U, D] \right)^2 = \sum_{i=1}^N (\delta H_i)^2 + \sum_{i<j}^N (H_i - H_j)^2 (U^\dagger \delta U)_{ij}^2 \quad (1.2)$$

so that the measure

$$dH = [dU] \prod_{i<j}^N (H_i - H_j)^2 \prod_{i=1}^N dH_i \quad (1.3)$$

where  $[dU]$  is the non-linear (Haar) measure for unitary matrices. Since the action in (1.1) does not depend on  $U$ , the integral  $V_N = \int [dU]$  fully decouples, and (1.1) turns into an  $N$ -fold integral over eigenvalues  $H_i$  with the peculiar square of the Van-der-Monde determinant  $\Delta(H) = \prod_{i>j}^N (H_i - H_j)$  in the measure:

$$Z_H(t) = V_N \int \prod_i dH_i \Delta^2(H) \exp \left( -\frac{1}{2g} \sum_i H_i^2 + \frac{1}{g} \sum_{i,k} t_k H_i^k \right) \quad (1.4)$$

### 1.3 Virasoro constraints

Integral (1.1) is invariant under any change of integration matrix-variable  $H$ . In particular,  $Z_H(t)$  does not change if one substitutes  $H \rightarrow H + \epsilon H^{n+1}$  with any matrix-valued parameter  $\epsilon$  and any integer  $n \geq -1$ . This invariance implies that [16]

$$\frac{\partial Z_H(t)}{\partial t_{n+2}} = \hat{L}_n Z_H(t), \quad n \geq -1 \quad (1.5)$$

with the operator  $\hat{L}_n$  defined in (4) and

$$\frac{\partial Z_H(t)}{\partial t_0} = \frac{N}{g} Z_H(t) \quad (1.6)$$

The l.h.s. in (1.5) is produced by the shift of the  $t_2$  variable  $t_2 \rightarrow t_2 - 1/2$  in the initial formula (1.1).

Together with (1.6) the system (1.5) provides a set of recurrent relations which allows one to unambiguously construct  $Z_H(t)$  term-by-term as a formal series in non-negative powers of  $t$ -variables.

### 1.4 Determinant representations and integrability

As we shall explain now, the properly normalized matrix integral  $\mathcal{Z}_N \equiv \frac{1}{V_N N!} Z_H(t)$  is a  $\tau$ -function of the Toda-chain integrable hierarchy [7, 15]. In this paragraph we rescale the time variables to cancel the coefficient  $g$  in front of them in order to have the standard definition of integrable hierarchies.

One of the technical ways to deal with integrals of form (1.4) was proposed in [3]. The authors introduced a system of orthogonal polynomials with the orthogonality condition

$$\int P_i(H)P_j(H)e^{-V(H)}dH = \delta_{ij}e^{\varphi_i(t)}, \tag{1.7}$$

where  $e^{\varphi_i(t)}$  are norms defined by integral (1.7) and the normalizing condition for the polynomials is

$$P_i(H) = \sum_{j \leq i} \gamma_{ij}H^j, \quad \gamma_{ii} = 1, \tag{1.8}$$

i.e., the coefficient of the leading term is put equal to unity.

Using polynomials (1.7), (1.8), we can rewrite (1.4) as

$$\begin{aligned} \mathcal{Z}_N &= (N!)^{-1} \int \prod_i dH_i \det P_{k-1}(H_j) \det P_{l-1}(H_m) \exp \left\{ -\frac{1}{2} \sum_i H_i^2 + \sum_{i,k} t_k H_i^k \right\} \\ &= \prod_{i=0}^{N-1} e^{\varphi_i(t)}. \end{aligned} \tag{1.9}$$

In order to get the determinant representation, we rewrite orthogonality condition (1.7) in a “matrix” form. That is, we introduce the matrix  $\Gamma$  with matrix elements  $\gamma_{mn}$  defined by (1.8), the so-called *moment matrix*  $C$  with the matrix elements

$$C_{ij} = \int dH H^i H^{j-2} e^{-V(H)}, \tag{1.10}$$

and the diagonal matrix  $J$  with diagonal elements  $e^{\varphi_n}$ . Then, (1.7) can be written as a matrix relation<sup>1</sup>

$$\Gamma C \Gamma^T = J \tag{1.11}$$

where  $\Gamma^T$  is the transposed matrix. Evaluating the determinant of the both sides of this relation and using (1.9), we obtain

$$\mathcal{Z}_N = \det_{N \times N} C_{ij}. \tag{1.12}$$

The moment matrix satisfies a number of relations, which follow directly from its explicit form (1.10),

$$\frac{\partial C_*(t)}{\partial t_k} \equiv \partial_k C_*(t) = \frac{\partial^k C_*(t)}{\partial t_1^k} \equiv \partial^k C_*(t), \tag{1.13}$$

$$C_{ij} = C_{i+j} \tag{1.14}$$

and

$$C_N = \partial^{N-2} C_{11} \equiv \partial^{N-2} C \tag{1.15}$$

Finally, the partition function of the one-matrix model is

$$\mathcal{Z}_N = \det \partial^{i+j-2} C, \tag{1.16}$$

---

<sup>1</sup>This relation is nothing but the Riemann-Hilbert problem also known as the factorization problem, see [15, 43, 44] for the details.

which results in the Toda chain [7, 15, 24, 40]. Note that conditions (1.13) and (1.15) are satisfied for the whole hierarchy of the two-dimensional Toda lattice and for the KP hierarchy, while (1.14) is specific for the Toda chain.

### 1.5 Kontsevich-Penner representation

The system (1.5) is solved by another integral, very different from (1.1) [45]:

$$Z_H(t) = \pi^{\frac{N^2-n^2}{2}} g^{\frac{(N-n)^2}{2}} i^{-Nn} e^{\frac{1}{2g} \text{tr} L^2} \int_{n \times n} \exp\left(-\frac{g}{2} \text{tr} h^2 + N \text{tr} \log h - i \text{tr} hL\right) dh \quad (1.17)$$

Integral is now over  $n \times n$  Hermitian matrix  $h$ ,  $dh = \prod_{a,b=1}^n dh_{ab}$  and depends on additional  $n \times n$  matrix (background field)  $L$ . To emphasize the difference between  $N$  and  $n$  we use small letters for  $h$  and  $\text{tr}$  instead of  $H$  and  $\text{Tr}$  in (1.1). If expanded around a saddle-point  $h = L$  this integral is a formal series in positive powers of variables

$$t_k = -\frac{g}{k} \text{tr} L^{-k}, \quad k \geq 1 \quad (1.18)$$

and<sup>2</sup>

$$t_0 = g \text{tr} \log L \quad (1.20)$$

This is a model from the GKM family and peculiar logarithmic term in the action is often named Penner term [46], so that (1.17) is known as *Gaussian Kontsevich-Penner model*.

#### 1.5.1 Proof I: Ward identities

In order to check that (1.17) satisfies (3) one begins with the Ward identity for this integral [23], associated to the shift  $h \rightarrow h + \epsilon$  of the integration variable by a small arbitrary matrix  $\epsilon$ :

$$\left(g \frac{\partial}{\partial L_{\text{tr}}} + N \left(\frac{\partial}{\partial L_{\text{tr}}}\right)^{-1} + L\right) \int_{n \times n} \exp\left(-\frac{g}{2} \text{tr} h^2 + N \text{tr} \log h - i \text{tr} hL\right) = 0 \quad (1.21)$$

which gives

$$\left(g \left(\frac{\partial}{\partial L_{\text{tr}}} - \frac{L}{g}\right)^2 + N + n + L \left(\frac{\partial}{\partial L_{\text{tr}}} - \frac{L}{g}\right)\right) Z = \left(g \frac{\partial^2}{\partial L_{\text{tr}}^2} + N - L \frac{\partial}{\partial L_{\text{tr}}}\right) Z = 0 \quad (1.22)$$

It remains to substitute a function  $Z_H(t)$  with  $t$ 's expressed through  $L$  by the *Miwa transform* (1.18). Then (1.21) becomes (1.5), see [7, 24, 45] for technical details.

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<sup>2</sup>It is possible to introduce dependence on  $t_0$  implicitly, namely to define the partition function as follows:

$$Z_H(t) = \pi^{\frac{N^2-n^2}{2}} g^{\frac{(N-n)^2}{2}} i^{-Nn} e^{\frac{t_0 N}{g} + \frac{1}{2g} \text{tr} L^2 - N \text{tr} \log L} \int_{n \times n} \exp\left(-\frac{g}{2} \text{tr} h^2 + N \text{tr} \log h - i \text{tr} hL\right) dh \quad (1.19)$$

this slightly change all calculations but leave the partition function unchanged.

### 1.5.2 Proof II: orthogonal polynomials

The two integral representations (1.1) and (1.17) can be related directly, without a reference to Virasoro constraints (1.5). One procedure, making use of orthogonal polynomials (Hermite polynomials in this particular case) is described in details in [7, 24]. One can rewrite (1.9) with time variables substituted by (1.18), (1.20) in terms of orthogonal polynomials in the following way:

$$\begin{aligned}
 \mathcal{Z}_N &= (N!)^{-1} \int \prod_i dH_i \Delta^2(H) \exp \left( - \sum_i \tilde{V}(H_i) + \frac{1}{g} \sum_{i,k=0} t_k H_i^k \right) = \\
 &= (N!)^{-1} \int \prod_i dH_i \Delta^2(H) \exp \left( - \sum_i \tilde{V}(H_i) \right) \prod_{i,a} (L_a - H_i) = \\
 &= (N!)^{-1} \int \prod_i dH_i \exp \left( - \sum_i \tilde{V}(H_i) \right) \Delta(H) \frac{\Delta(H, L)}{\Delta(L)} = \\
 &= (N!)^{-1} \Delta^{-1}(L) \int \prod_i dH_i \exp \left( - \sum_i \tilde{V}(H_i) \right) \times \\
 &\quad \times \det_{N \times N} \tilde{P}_{i-1}(H_j) \det_{(N+n) \times (N+n)} \begin{bmatrix} \tilde{P}_{i-1}(H_j) : \tilde{P}_{N+b-l}(H_j) \\ \dots : \dots \\ \tilde{P}_{i-1}(L_a) : \tilde{P}_{N+b-l}(L_a) \end{bmatrix},
 \end{aligned}$$

where  $i, j = 1, \dots, N$ ,  $a, b = 1, \dots, n$  and the orthogonal polynomials  $\tilde{P}_k(H)$  are orthogonal with the measure  $\exp(-\tilde{V}(H))$ . In the case under consideration, it is equal to  $\exp\left(-\frac{1}{2g}H^2\right)$ . Now calculating the determinants and using the orthogonality condition (1.7), one arrives at

$$\begin{aligned}
 \mathcal{Z}_N &= \Delta^{-1}(L) \det_{n \times n} \tilde{P}_{N+a-1}(L_b) \prod_i e^{\tilde{\varphi}_i(s)} = \\
 &= \left[ \prod_i e^{\tilde{\varphi}_i(s)} \right] \frac{\det_{(ab)} \phi_a^{(N)}(L_b)}{\Delta(L)} = \mathcal{Z}_N|_{L_a=\infty} \times \frac{\det_{(ab)} \phi_a^{(N)}(L_b)}{\Delta(L)}, \quad (1.23)
 \end{aligned}$$

with

$$\phi_a^{(N)}(L) = \tilde{P}_{N+a-1}(L) \quad (1.24)$$

Now let us see that integral (1.17) can be also transformed to this form. To this end, we need the Itzykson-Zuber formula, [47]

$$\int_{U(n)} dU e^{\text{tr } AUBU^\dagger} = V_n \frac{\det e^{a_i b_j}}{\Delta(x)\Delta(y)} \quad (1.25)$$

where integral runs over unitary  $n \times n$  matrices with the Haar measure  $dU$ , and  $a_i, b_j$  are eigenvalues of Hermitian matrices  $A$  and  $B$ . Now, using (1.3), we can perform integration

over the angular variables and rewrite (1.17) as

$$\begin{aligned}
 e^{\frac{1}{2g} \text{tr } L^2} \int_{n \times n} dh \exp \left( -\frac{g}{2} \text{tr } h^2 + N \text{tr } \log h - i \text{tr } hL \right) &\sim \\
 \sim e^{\frac{1}{2g} \sum_i L_i^2} \int \prod_i dh_i \frac{\Delta(h)}{\Delta(L)} \exp \left( -\frac{g}{2} \sum_i h_i^2 + N \sum_i \log h_i - i \sum_i h_i L_i \right) &= \\
 = e^{\frac{1}{2g} \sum_i L_i^2} \int \prod_i dh_i \frac{\det h_i^{j-1}}{\Delta(L)} \exp \left( -\frac{g}{2} \sum_i h_i^2 + N \sum_i \log h_i - i \sum_i h_i L_i \right) &= \frac{\det \Phi_i(L_j)}{\Delta(L)}
 \end{aligned} \tag{1.26}$$

where

$$\Phi_i(L) \equiv e^{\frac{1}{2g} L^2} \int dx x^{i-1} \exp \left( -\frac{g}{2} x^2 + N \log x - ixL \right) \tag{1.27}$$

It remains to note that the orthogonal polynomials with the measure  $\exp \left( -\frac{1}{2g} H^2 \right)$  are the Hermit polynomials which have the integral representation

$$\tilde{P}_k(x) = \frac{g^{k/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy \left( \frac{x}{\sqrt{g}} + iy \right)^k e^{-y^2/2} = \frac{i^n g^{k+\frac{1}{2}}}{\sqrt{2\pi}} e^{\frac{x^2}{2g}} \int_{-\infty}^{+\infty} dy y^k e^{-\frac{gy^2}{2} - ixy} \tag{1.28}$$

This finally reduces (1.23) to (1.17).

### 1.5.3 Proof III: Faddeev-Popov trick

Another way to connect the two matrix integrals, suggested recently in [21] is by using the Faddeev-Popov trick. In order not to make calculations with Grassmann variables, we choose the opposite sign in (1.18), (1.20)

$$t_k = \frac{g}{k} \text{tr } L^{-k}, \quad k \geq 1 \tag{1.29}$$

$$t_0 = -g \text{tr } \log L \tag{1.30}$$

Then, (1.17) should be substituted with (the results for the two choices of sign can be also related by continuation)

$$\sim e^{-\frac{1}{2g} \text{tr } L^2} \int_{n \times n} \exp \left( -\frac{g}{2} \text{tr } h^2 - N \text{tr } \log h + \text{tr } hL \right) dh \tag{1.31}$$

If Miwa transform (1.29) is made in the original integral (1.1), it becomes [7]

$$\begin{aligned}
 (1.1) &= (\det L)^{-N} \int_{N \times N} \frac{e^{-\frac{1}{2g} \text{Tr } H^2} dH}{\det(I \otimes I - H \otimes L^{-1})} \\
 &= \int \int \int e^{-\frac{1}{2g} \text{Tr } H^2 + B(I \otimes L - H \otimes I)C} dH d^2 B,
 \end{aligned} \tag{1.32}$$

where Faddeev-Popov trick is applied to substitute the determinant in the denominator by an integral over auxiliary rectangular  $N \times n$  complex matrix fields  $B$  and  $C = B^\dagger$ . Here  $d^2 B = dBdC = \prod_{i=1}^N \prod_{a=1}^n d^2 B_{ia}$  and

$$B(H \otimes I - I \otimes L)C = B_{ai} H_{ij} C_{ja} - B_{ia} L_{ab} C_{bi} = \text{tr } B^\dagger H B - \text{Tr } B L B^\dagger \tag{1.33}$$

Taking the Gaussian integral over  $H$  we finally obtain:

$$(1.1) = g^{\frac{N^2}{2}} \int \int e^{\frac{g}{2} \text{Tr } BB^\dagger BB^\dagger + \text{Tr } BLB^\dagger} d^2 B \tag{1.34}$$

At the same time the Kontsevich-Penner integral (1.31) is equal to

$$\begin{aligned} (1.31) &= g^{\frac{N^2}{2} + \frac{n^2}{2}} \int \frac{e^{-\frac{g}{2} \text{tr } h^2} dh}{\det(L + gh)^N} = g^{\frac{N^2}{2} + \frac{n^2}{2}} \int \frac{e^{-\frac{g}{2} \text{tr } h^2} dh}{\det(I \otimes gh + I \otimes L)} = \\ &= g^{\frac{N^2}{2} + \frac{n^2}{2}} \int \int \int e^{-\frac{g}{2} \text{tr } h^2 + B(I \otimes gh + I \otimes L)C} dh dB dC \end{aligned}$$

where the fields  $B, C$  are exactly the same as in (1.32), while

$$B(I \otimes gh + I \otimes L)C = gB_{ia}h_{ab}C_{bi} + B_{ia}L_{ab}C_{bi} = g\text{Tr } BhB^\dagger + \text{Tr } BLB^\dagger \tag{1.35}$$

Again we can take the Gaussian integral over  $h$  and obtain:

$$(1.31) = g^{\frac{N^2}{2}} \int \int e^{\frac{g}{2} \text{Tr } BB^\dagger BB^\dagger + \text{Tr } BLB^\dagger} d^2 B \tag{1.36}$$

i.e. exactly the same expression as at the r.h.s. of (1.34). Thus we conclude that

$$(1.1) = (1.31) \tag{1.37}$$

the two integral representation for  $Z_H(t)$  coincide.

Inverting the argument, the two matrix-integral representations (1.1) and (1.31) for  $Z_H(t)$  are associated with two ways to decompose the quartic vertex  $\text{Tr } BB^\dagger BB^\dagger = \text{tr } B^\dagger BB^\dagger B$  with the help of auxiliary fields  $H$  and  $h$ , coupled respectively to  $BB^\dagger$  and  $B^\dagger B$  and thus having the different sizes:  $N \times N$  and  $n \times n$ .

As a word of precaution we remind only that for any finite  $n$  the Miwa transform (1.18) defines only an  $n$ -dimensional *subset* in the infinite-dimensional space of  $t$ -variables: when expressed through  $L$  the higher  $t_k$  with  $k > n$  are actually algebraic functions of the lowest  $t_1, \dots, t_n$ . Thus  $Z_H(t)$  in this context should be interpreted as a projective limit at  $n \rightarrow \infty$ .

### 1.6 Genus expansion and the first multiresolvents

Multiresolvents for Hermitian model are described in detail in the reference-paper [26]. Here we remind only the simplest of the relevant formulas.

Multiresolvents are defined by eq. (10) and the first step is to rewrite Virasoro constraints as recurrent relations for  $\rho^{(p|q)}$ . Such recursive reformulation is possible only if the *genus expansion* of the *free energy*  $\mathcal{F} = \log Z$  is performed,  $\mathcal{F} = \sum_{p=0}^{\infty} g^{2p-2} \mathcal{F}^{(p)}$ . As explained in some detail in [26], this requirement picks up some rather special solutions to Virasoro constraints, and only such solutions possess well-defined multiresolvents and are associated with the *bare spectral curves* of finite genera.<sup>3</sup> The bare spectral curve  $\Sigma$

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<sup>3</sup>It deserves emphasizing that *genus*  $p$  in "genus expansion" refers to the genus of the fat-graph Feynman diagrams contributing to  $\mathcal{F}^{(p)}$ . It has nothing to do with the genus of the bare spectral curve, throughout this text *this* genus will be only zero, while the genus of the *full* spectral curve (which defines the point of the Universal Grassmannian [41] underlying the matrix-model  $\tau$ -function) is infinite. Relation between bare and full spectral curves is rather tricky and is not yet fully clarified in the literature.

is defined from non-linear equation for  $\rho^{(0|1)}$  — the starting point of the recursion. The next step provides  $\rho^{(0|2)}$ , which appears to be easily connected with Bergman kernel bi-differential on  $\Sigma$  [48]. At each step of recursion there exists a certain arbitrariness in the choice of solutions, which is, however, absent for the Virasoro constraint (1.5) — crucial for this *unambiguity* is the form of the l.h.s. of (1.5): the fact that it is obtained by the *shift* of the time variable  $t_2 \rightarrow t_2 - 1/2$ . *Shift* is parameter which is allowed to stand in denominators when we build up a formal-series solution to Virasoro constraints. The shift of  $t_2$  is obviously associated with the integral (1.1) and defines what is naturally called the *Gaussian* phase of Hermitian model. If other time or many times are shifted, then arbitrariness is unavoidable, see [49] for its full description. If partly fixed and parameterized by several arbitrary variables, it provides the family of Dijkgraaf-Vafa non-Gaussian partition functions [48, 50, 51].<sup>4</sup>

**Gaussian phase.** The bare spectral curve for Gaussian phase of Hermitian model is

$$\Sigma_{\text{HG}} : y^2 = z^2 - 4S \tag{1.38}$$

where  $S = gN$  and a few first Gaussian multiresolvents are:

$$\rho_H^{(0|1)} = \frac{z - \sqrt{z^2 - 4S}}{2} \tag{1.39}$$

$$\rho_H^{(0|2)}(z_1, z_2) = \frac{1}{2(z_1 - z_2)^2} \left( \frac{z_1 z_2 - 4S}{y(z_1)y(z_2)} - 1 \right) \tag{1.40}$$

$$\rho_H^{(1|1)}(z) = \frac{S}{y^5(z)} \tag{1.41}$$

$$\rho_H^{(0|3)}(z_1, z_2, z_3) = \frac{2S(z_1 z_2 + z_2 z_3 + z_3 z_1 + 4S)}{y^3(z_1)y^3(z_2)y^3(z_3)} \tag{1.42}$$

$$\rho_H^{(1|2)}(z_1, z_2) = \frac{S}{y^7(z_1)y^7(z_2)} \left( z_1 z_2 (5z_1^4 + 4z_1^3 z_2 + 3z_1^2 z_2^2 + 4z_1 z_2^3 + 5z_2^4) + \right. \tag{1.43}$$

$$\left. + 4S(z_1^4 - 13z_1 z_2(z_1^2 + z_1 z_2 + z_2^2) + z_2^4) + 16S^2(-z_1^2 + 13z_1 z_2 - z_2^2) + 320S^3 \right)$$

$$\rho_H^{(2|1)}(z) = \frac{21S(z^2 + S)}{y^{11}(z)} \tag{1.44}$$

---

<sup>4</sup>Relation between generic genus-expansion-possessing solutions of [49] and the Dijkgraaf-Vafa family is very similar to that between the "general" and "total" solutions to the Hamilton-Jacobi equation, see [52, sect.7].

They are deduced from the recurrent relations

$$y(z_1)\rho_H^{(0|2)}(z_1, z_2) = \partial_{z_2} \frac{\rho_H^{(0|1)}(z_1) - \rho_H^{(0|1)}(z_2)}{z_1 - z_2} \quad (1.45)$$

$$y(z_1)\rho_H^{(1|1)}(z_1) = \rho_H^{(0|2)}(z_1, z_1) \quad (1.46)$$

$$y(z_1)\rho_H^{(0|3)}(z_1, z_2, z_3) = 2\rho_H^{(0|2)}(z_1, z_2)\rho_H^{(0|2)}(z_1, z_3) + \partial_{z_2} \frac{\rho_H^{(0|2)}(z_1, z_3) - \rho_H^{(0|2)}(z_2, z_3)}{z_1 - z_2} + \partial_{z_3} \frac{\rho_H^{(0|2)}(z_1, z_2) - \rho_H^{(0|2)}(z_2, z_3)}{z_1 - z_3} \quad (1.47)$$

$$y(z_1)\rho_H^{(1|2)}(z_1, z_2) = 2\rho_H^{(0|2)}(z_1, z_2)\rho_H^{(1|1)}(z_1) + \rho_H^{(0|3)}(z_1, z_1, z_2) + \partial_{z_2} \frac{\rho_H^{(1|1)}(z_1) - \rho_H^{(1|1)}(z_2)}{z_1 - z_2} \quad (1.48)$$

$$y(z_1)\rho_H^{(2|1)}(z_1) = \left(\rho_H^{(1|1)}(z_1)\right)^2 + \rho_H^{(1|2)}(z_1, z_1) \quad (1.49)$$

**Non-Gaussian phases.** For the sake of completeness we also give some formulas for non-Gaussian phases. If instead of the Gaussian shift  $t_2 \rightarrow t_2 - \frac{1}{2}$  we apply  $t_k \rightarrow t_k - T_k$  with  $W(z) = \sum_{k=0}^{n+1} T_k z^k$  and rewrite the shifted Virasoro constraints in terms of the multiresolvents we get:

$$W'(z)\rho(z) = \rho^2(z) + f(z) + g^2 \hat{\nabla}(z)\rho(z) + \hat{P}_z^- [v'(z)\rho(z)] \quad (1.50)$$

where

$$\hat{\nabla}(z) = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \frac{\partial}{\partial t_k} \quad (1.51)$$

and

$$\rho(z) = \hat{\nabla}(z)g^2 \log Z \quad (1.52)$$

$$f(z) = \hat{P}_z^+ [W'(z)\rho(z)] = \hat{R}_H(z)g^2 \log Z \quad (1.53)$$

$$\begin{aligned} & W'(z)\rho_W^{(p|m+1)}(z, z_1, \dots, z_m) - f_W^{(p|m+1)}(z|z_1, \dots, z_m) = \\ & = \sum_q \sum_{m_1+m_2=m} \rho_W^{(q|m_1+1)}(z, z_{i_1}, \dots, z_{i_{m_1}})\rho_W^{(p-q|m_2+1)}(z, z_{j_1}, \dots, z_{j_{m_2}}) + \\ & + \sum_{i=1}^m \frac{\partial}{\partial z_i} \frac{\rho_W^{(p|m)}(z, z_1, \dots, \check{z}_i, \dots, z_m) - \rho_W^{(p|m)}(z_1, \dots, z_m)}{z - z_i} + \hat{\nabla}(z)\rho_W^{(p-1|m+1)}(z, z_1, \dots, z_m) \end{aligned} \quad (1.54)$$

$$f_H^{(p|m+1)}(z|z_1, \dots, z_m) = \check{R}_H(z)\rho^{(p|m)}(z_1, \dots, z_m) \quad (1.55)$$

$$\check{R}_H(z) = P_z^+ [W'(z)\check{\nabla}(z)] = - \sum_{a=0}^{n-1} \sum_{b=0}^{n-a-1} (a+b+2)T_{a+b+2}z^a \frac{\partial}{\partial T_b} \quad (1.56)$$

For further details we refer to [26, 49] and references therein.



## 2 Complex-matrix model $Z_C(t)$

### 2.1 Original integral representation

Complex matrix model was originally defined as an integral over  $N \times N$  complex matrices  $\Phi$

$$Z_C(t) = \int_{N \times N} \exp \left( -\frac{\text{Tr} \Phi \Phi^\dagger}{g} + \sum_{k=0}^{\infty} \frac{t_k}{g} \text{Tr} (\Phi \Phi^\dagger)^k \right) d^2\Phi \quad (2.1)$$

where  $d^2\Phi = \left(\frac{i}{2}\right)^{N^2} \prod_{i,j=1}^N d^2\Phi_{ij} = \prod_{i,j=1}^N d\text{Re}(\Phi_{ij})d\text{Im}(\Phi_{ij}) = \prod_{i,j=1}^N \frac{i}{2}d\Phi_{ij}d\Phi_{ij}^\dagger$ .

### 2.2 Eigenvalue representation

One can express a complex matrix  $\Phi$  through Hermitian  $H$  and unitary  $U$  matrices,

$$\Phi = UH, \quad \Phi^\dagger = HU^\dagger \quad (2.2)$$

and, further, through diagonal matrix  $D$  and two unitary matrices  $U$  and  $V$ :

$$\Phi = UDV^\dagger, \quad \Phi^\dagger = VDU^\dagger \quad (2.3)$$

The norm of  $\Phi$  decomposes as

$$\begin{aligned} \text{Tr} \delta\Phi\delta\Phi^\dagger &= \text{Tr} (\delta H)^2 - \text{Tr} H^2(U^\dagger\delta U)^2 + \text{Tr} [H, \delta H](U^\dagger\delta U) = \\ &= \text{Tr} (\delta D)^2 - \text{Tr} (U^\dagger\delta U)^2 D^2 - \text{Tr} D^2(V^\dagger\delta V)^2 + 2\text{Tr} (U^\dagger\delta U)D(V^\dagger\delta V)D, \end{aligned} \quad (2.4)$$

so that the measure

$$d^2\Phi = [dU][dV] \prod_{i<j}^N (H_i^2 - H_j^2)^2 \prod_{i=1}^N H_i dH_i \quad (2.5)$$

where  $H_i$  are the eigenvalues of matrix  $H$  (i.e. the entries of  $D$ ). In particular, for  $N = 1$  we have  $\Phi = e^{i\theta}H$  and  $d^2\Phi = HdHd\theta$ .

Comparing with (1.2), (1.3), one can see that for complex matrices the measure is actually the same as for Hermitian matrix  $H^2$ , such that  $\Phi\Phi^\dagger = UH^2U^\dagger$ . Since action in the model (2.1) also respects this substitution, we obtain:

$$\begin{aligned} Z_C(t) &= V_N^2 \int_0^\infty \prod_i dH_i^2 \Delta^2(H^2) \exp \left( -\frac{1}{g} \sum_i H_i^2 + \frac{1}{g} \sum_{i,k} t_k H_i^{2k} \right) \sim \\ &\sim \int [dU] \int \exp \left( -\frac{1}{g} \text{Tr} H^2 + \frac{1}{g} \sum_{k=0}^{\infty} t_k \text{Tr} H^{2k} \right) d(H^2) \end{aligned}$$

This integral looks just the same as (1.1) for Hermitian matrix model, however, there is a difference in the integration contour:  $H^2$  is not an arbitrary Hermitian matrix. Relation between (1.1) and (2.6) is like between an integral over entire real axis and over its positive ray: the answers are different and even invariance properties — and thus the Picard-Fuchs equations (Ward identities) are not exactly the same.

**2.3 Virasoro constraints**

Since the eq. (2.6) means that the Virasoro constraints are the same as the "discrete Virasoro constraints" for Hermitian matrix model. However, there are two differences.

First,  $L_{-1}$ -constraint, associated with the shift  $\delta(H^2) = \epsilon$ , is excluded, because it would correspond to a singular transform  $\delta H \sim H^{-1}$ . This exclusion can also be considered as a result of the above-mentioned change of integration contour: from entire real line in the case of Hermitian model to a positive ray  $0 \leq H^2 < \infty$  in the case of (2.6).

Second, the shift of time variables is  $t_k = \tilde{t}_k - \frac{1}{2}\delta_{k,2}$  in the Gaussian phase of Hermitian model, but it is rather  $t_k = \tilde{t}_k - \delta_{k,1}$  in the Gaussian phase of (2.6).

The two changes together make the seemingly diminished set of Virasoro constraints

$$\hat{L}_m Z_C = 0, \quad m \geq 0, \tag{2.6}$$

with

$$L_m(t) = -\frac{\partial}{\partial t_{1+m}} + \sum_{k \geq 1} k t_k \frac{\partial}{\partial t_{k+m}} + g^2 \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b} \tag{2.7}$$

and

$$\frac{\partial Z_C}{\partial t_0} = \frac{S}{g^2} Z_C \tag{2.8}$$

fully exhaustive: despite the lack of  $\hat{L}_{-1}$ , these equations are enough for *unambiguous* recursive reconstruction of all terms in the formal series  $Z_C(t)$  for the Gaussian branch of the complex matrix model.

**2.4 Determinant representation and integrability**

The integrable properties of the complex matrix model are practically identical to those of the Hermitian model. In particular, the partition function

$$Z_C \sim \mathcal{Z}_N^C = \det \partial^{i+j-2} C \tag{2.9}$$

where the moment matrix is given a bit different integral as compared with the Hermitian case,

$$C_{ij} \equiv \int_0^\infty dx \exp \left( -\frac{1}{g} x + \frac{1}{g} \sum_k t_k x^k \right) \tag{2.10}$$

This is still a Toda chain  $\tau$ -function, however, it corresponds to another solution to the hierarchy given by the Virasoro constraints (2.6).

**2.5 Kontsevich-Penner representation**

Like in the case of Hermitian model, the set of constraints (2.6) has another matrix-integral solution [31], different from (2.1):

$$Z_C(t) = \pi^{N^2-n^2} e^{-\text{tr } \eta^\dagger} \int \exp \left( -\text{tr } \phi \phi^\dagger + \text{tr } \eta^\dagger \phi + \text{tr } \phi^\dagger \eta + N \text{tr } \log(\phi \phi^\dagger) \right) d^2 \phi \tag{2.11}$$

This time integral is over complex matrices  $\phi$ , but their size  $n$  is, like in (1.17), independent of  $N$ , which appears only as a parameter in the Penner term. For the sake of simplicity, we

put here  $g = 1$ , the  $g$ -dependence being easily restorable. The time variables are related to the external matrices  $\eta$  and  $\eta^\dagger$  as

$$\begin{aligned} t_k &= -\frac{1}{k} \text{tr} (\eta\eta^\dagger)^{-k}, \quad k \geq 1, \\ t_0 &= \log(\eta\eta^\dagger) \end{aligned} \quad (2.12)$$

### 2.5.1 Proof I: Ward identities

The Ward identity associated with the shifts  $\phi \rightarrow \phi + \epsilon$  of the integration variable, is now

$$\left[ -\frac{\partial}{\partial \eta_{\text{tr}}} + N \left( \frac{\partial}{\partial \eta_{\text{tr}}^\dagger} \right)^{-1} + \eta^\dagger \right] \int \exp \left( -\text{tr} \phi\phi^\dagger + \text{tr} \eta^\dagger \phi + \text{tr} \phi^\dagger \eta + N \text{tr} \log(\phi\phi^\dagger) \right) d^2\phi = 0 \quad (2.13)$$

Therefore, one gets

$$\begin{aligned} \left[ -\left( \frac{\partial}{\partial \eta_{\text{tr}}^\dagger} + \eta \right) \left( \frac{\partial}{\partial \eta_{\text{tr}}} + \eta^\dagger \right) + N + \left( \frac{\partial}{\partial \eta_{\text{tr}}^\dagger} + \eta \right) \eta^\dagger \right] Z_C = \\ = \left[ -\frac{\partial^2}{\partial \eta \partial \eta^\dagger} + N - \frac{1}{2} \eta \frac{\partial}{\partial \eta} - \frac{1}{2} \eta^\dagger \frac{\partial}{\partial \eta^\dagger} \right] Z_C = 0 \end{aligned} \quad (2.14)$$

and, substituting  $Z_C$  as a function of the Miwa variables (2.12) one reproduces (2.6), see [31].

### 2.5.2 Proof II: orthogonal polynomials

Let us put  $L \equiv \eta\eta^\dagger$ . Then, one can immediately repeat the calculation of section 1.5.2 in order to obtain that (2.1) is equal to (1.23) and (1.24), where the polynomials  $\tilde{P}_k(H)$  are now orthogonal with the weight  $\exp(-x)$  on the positive real semi-axis. Such orthogonal polynomials are nothing but the Laguerre polynomials [53], which have the following integral representation

$$\tilde{P}_k(x) = (-1)^k e^x \int_0^\infty dy y^k e^{-y} J_0(2\sqrt{xy}) \quad (2.15)$$

where  $J_0(x)$  is the zero order Bessel function.

Now let us rewrite (2.11) in the determinant form. This time in order to integrate over angular variables, we need to use instead of the Itzykson-Zuber formula the following very nice formula of integration over two unitary  $n \times n$  matrices, [54]

$$\int_{U(n)} dU \int_{U(n)} dV \exp \left( \frac{1}{2} \text{tr} \left[ U A V B + B^\dagger V^\dagger A^\dagger U^\dagger \right] \right) = 2^{n(n-1)} V_n^2 \frac{\det J_0(x_i y_j)}{\Delta(x^2) \Delta(y^2)} \quad (2.16)$$

where  $x_i^2$  and  $y_j^2$  are the eigenvalues of  $A^\dagger A$  and  $B B^\dagger$  respectively,  $A$  and  $B$  being arbitrary  $n \times n$  complex matrices. Using this formula and (2.5) and denoting eigenvalues of  $\phi\phi^\dagger$  and  $\eta\eta^\dagger$  through  $y_i$  and  $x_i$  respectively, one obtains

$$\begin{aligned} e^{-\text{tr} \eta\eta^\dagger} \int \exp \left( -\text{tr} \phi\phi^\dagger + \text{tr} \eta^\dagger \phi + \text{tr} \phi^\dagger \eta + N \text{tr} \log(\phi\phi^\dagger) \right) d^2\phi \sim \\ \sim e^{-\sum_i x_i} \int_0^\infty \prod_i dy_i \frac{\Delta(y)}{\Delta(x)} \exp \left( -\sum_i y_i + N \sum_i \log y_i \right) J_0(2\sqrt{x_i y_i}) = \frac{\det \Phi_i^{(C)}(x_j)}{\Delta(x)} \end{aligned} \quad (2.17)$$

where

$$\Phi_i^{(C)}(x) = e^{-x} \int_0^\infty dy y^{i-1} \exp(-y + N \log y J_0(2\sqrt{xy})) \quad (2.18)$$

Comparing this with (2.15) we ultimately identify (2.1) and (2.11).

### 2.5.3 Proof III: Faddeev-Popov trick

Direct equivalence of the two integrals (2.1) and (2.11) can be proved by a somewhat tricky generalization of Faddeev-Popov argument from [21], which we applied in section 1.5 above.

As before, we make the other choice of the sign in the Miwa transform,

$$t_k = \frac{1}{k}(\eta\eta^\dagger)^{-k}, \quad k \geq 1 \quad (2.19)$$

in order to deal with bosonic auxiliary fields. After the Miwa transform, the integral (2.1) becomes

$$\begin{aligned} (2.1) &= \int \exp\left(-\text{Tr } \Phi\Phi^\dagger + \sum_{k=0}^{\infty} \frac{1}{k} \text{tr } (\eta\eta^\dagger)^{-k} \text{Tr } (\Phi\Phi^\dagger)^k\right) d^2\Phi \\ &= \det(-\eta\eta^\dagger)^N \int \frac{e^{-\text{Tr } \Phi\Phi^\dagger} d^2\Phi}{\det(\Phi\Phi^\dagger \otimes I - I \otimes \eta\eta^\dagger)} = \\ &= \det(-\eta\eta^\dagger)^N \int \int e^{-\text{Tr } \Phi\Phi^\dagger + \text{tr } B^\dagger\Phi\Phi^\dagger B - \text{Tr } B\eta\eta^\dagger B^\dagger} d^2\Phi d^2B = \\ &= \det(-\eta\eta^\dagger)^N \int \frac{e^{-\text{tr } \eta\eta^\dagger B^\dagger B} d^2B}{\det_{N \times N}(I - BB^\dagger)^N} \end{aligned} \quad (2.20)$$

Note that the last determinant is raised to the power  $N$ , this is because we integrate over  $N^2$  complex-valued variables  $\Phi_{ij}$ : the relevant piece of the action is  $\sum_{i,j,k=1}^N \Phi_{ij} \bar{\Phi}_{ik} (\delta_{jk} - \sum_{a=1}^n B_{ja} \bar{B}_{ka})$ .

At the same time integral (2.11) is:

$$\begin{aligned} (2.11) &= \int \frac{e^{-\text{tr } \phi\phi^\dagger} d^2\phi}{\det_{n \times n}^N((\phi + \eta)(\phi^\dagger + \eta^\dagger))} = \int \int e^{-\text{tr } \phi\phi^\dagger - \text{Tr } b^\dagger(\phi - \eta)(\phi^\dagger - \eta^\dagger)b} d^2\phi d^2b \\ &= \int \frac{e^{-\text{tr } \eta\eta^\dagger \frac{b^\dagger b}{1+b^\dagger b}} d^2b}{\det_{n \times n}(1 + b^\dagger b)^n} \end{aligned} \quad (2.21)$$

Like  $B$ ,  $b$  is rectangular  $N \times n$  matrix, but the integrals are not literally equal as it was in the case of Hermitian model: one still needs to relate  $B$  and  $b$ .

Let us begin with a few examples.

**Examples.**  $N = n = 1$ : In this case we can introduce new variables:  $\rho = B^\dagger B = |B|^2$  and  $\sigma = b^\dagger b = |b|^2$ . Denoting also  $K = \eta\eta^\dagger = |\eta|^2$ , we obtain our two integrals in the form:

$$N = n = 1: \quad (2.1) = \int \frac{e^{-\rho K} d\rho}{1 - \rho}, \quad \text{while} \quad (2.11) = \int \frac{e^{-\frac{\sigma}{1+\sigma} K} d\sigma}{1 + \sigma} \quad (2.22)$$

and integrands coincide because for  $\rho = \frac{\sigma}{1+\sigma}$  we have  $d\rho = -\frac{d\sigma}{(1+\sigma)^2}$ , while  $1 - \rho = \frac{1}{1+\sigma}$ .

$n = 1, N$  arbitrary: In this case  $B$  is a complex  $N$ -vector  $(B_1, \dots, B_N)$ , the  $N \times N$  matrix  $BB^\dagger$  has rank 1 and  $\det(I - BB^\dagger) = 1 - |B_1|^2 - \dots - |B_N|^2 = 1 - \rho_1 - \dots - \rho_N = 1 - \rho_+$ , so that

$$(2.1) = \int \frac{e^{-\rho_+ K} d\rho_1 \dots d\rho_N}{(1 - \rho_+)^N} \sim \int \frac{\rho_+^{N-1} e^{-\rho_+ K} d\rho_+}{(1 - \rho_+)^N} \quad (2.23)$$

and similarly

$$(2.11) = \int \frac{e^{-\frac{\sigma_+}{1+\sigma_+} K} d\sigma_1 \dots d\sigma_N}{1 + \sigma_+} \sim \int \frac{\sigma_+^{N-1} e^{-\frac{\sigma_+}{1+\sigma_+} K} d\sigma_+}{1 + \sigma_+}, \quad (2.24)$$

where we used the fact that the volume of a simplex  $\rho_1 > 0, \dots, \rho_N > 0, \rho_1 + \dots + \rho_N = \rho_+$  is proportional to  $\rho_+^{N-1}$  and similarly for  $\sigma$ 's. Making the same transformation as above,  $\rho_+ = \frac{\sigma_+}{1+\sigma_+}$ , we see again that the integrals coincide.

$N = 1, n$  arbitrary: This time  $B$  and  $b$  are complex  $n$ -vectors. If we perform an  $SU(n)$  rotation to diagonalize  $\eta\eta^\dagger \rightarrow \text{diag}(K_1, \dots, K_n)$ , then both integrals (2.1) and (2.11) still contain  $B$  and  $b$  only in the form of squared modules  $\rho_a = |B_a|^2$  and  $\sigma_a = |b_a|^2$ :

$$(2.1) = \int \frac{e^{-\sum_{a=1}^n \rho_a K_a} d\rho_1 \dots d\rho_n}{1 - \rho_+} \quad \text{and} \quad (2.11) = \int \frac{e^{-\sum_{a=1}^n \sigma_a K_a / (1+\sigma_+)} d\sigma_1 \dots d\sigma_n}{(1 + \sigma_+)^n} \quad (2.25)$$

The integrals are related by our usual change of variables  $\rho_a = \frac{\sigma_a}{1+\sigma_+}$ , only the measure transform gets a little trickier:

$$\begin{aligned} \wedge_{a=1}^n d\rho_a &= \wedge_{a=1}^n \left( \frac{d\sigma_a}{1 + \sigma_+} - \frac{\sigma_a d\sigma_+}{(1 + \sigma_+)^2} \right) = \frac{\wedge_{a=1}^n d\sigma_a}{(1 + \sigma_+)^n} - \frac{\sum_{a=1}^n \sigma_a \wedge_{a=1}^n d\sigma_a}{(1 + \sigma_+)^{n+1}} \\ &= \frac{1}{(1 + \sigma_+)^{n+1}} \wedge_{a=1}^n d\sigma_a \end{aligned} \quad (2.26)$$

Substituting also  $1 - \rho_+ = (1 - \sigma_+)^{-1}$ , we see that the two integrals are in fact the same.

## 2.6 Genus expansions and the first multiresolvents

**Gaussian phase.** The first resolvent [26] in this case is

$$\rho_C^{(0|1)}(z) = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4S}{z}} \right) \quad (2.27)$$

and the bare spectral curve

$$\Sigma_C : \quad y^2 = z(z - 4S) \quad (2.28)$$

A few next multiresolvents are:

$$\rho_C^{(0|2)}(z_1, z_2) = \frac{1}{2(z_1 - z_2)^2} \left( \frac{z_1 z_2 - 2S(z_1 + z_2)}{y_1 y_2} - 1 \right) \quad (2.29)$$

$$\rho_C^{(1|1)}(z) = \frac{zS^2}{y_C(z)^5} \quad (2.30)$$

$$\rho_C^{(0|3)}(z_1, z_2, z_3) = 2 \frac{z_1 z_2 z_3 S^2}{y_C(z_1)^3 y_C(z_2)^3 y_C(z_3)^3} \quad (2.31)$$

$$\rho_C^{(1|2)}(z_1, z_2) = \frac{z_1^2 z_2^2 S^2}{y_1^2 y_2^2} (3z_2 z_1^3 + 3z_2^3 z_1 + 2z_2^2 z_1^2 - 2(z_1^3 + 17z_1^2 z_2 + 17z_1 z_2^2 + z_2^3)S + 8(3z_1^2 + 19z_1 z_2 + 3z_2^2)S^2 - 96(z_1 + z_2)S^3 + 128S^4) \quad (2.32)$$

$$\rho_C^{(2|1)}(z) = \frac{(9S^2 - 8zS + 8z^2)S^2 z^3}{y(z)^{11}} \quad (2.33)$$

They are deduced from the recurrent relations

$$\frac{y(z_1)}{z_1} \rho_C^{(0|2)}(z_1, z_2) = \frac{1}{z_1} \partial_{z_2} \frac{z_1 \rho_C^{(0|1)}(z_1) - z_2 \rho_C^{(0|1)}(z_2)}{z_1 - z_2} \quad (2.34)$$

$$\frac{y(z)}{z} \rho_C^{(1|1)}(z) = \rho_C^{(0|2)}(z, z) \quad (2.35)$$

$$\begin{aligned} \frac{y(z_1)}{z_1} \rho_C^{(0|3)}(z_1, z_2, z_3) &= 2\rho_C^{(0|2)}(z_1, z_2)\rho_C^{(0|2)}(z_1, z_3) + \frac{1}{z_1} \partial_{z_2} \frac{z_1 \rho_C^{(0|2)}(z_1, z_3) - z_2 \rho_C^{(0|2)}(z_2, z_3)}{z_1 - z_2} + \\ &+ \frac{1}{z_1} \partial_{z_3} \frac{z_1 \rho_C^{(0|2)}(z_1, z_2) - z_3 \rho_C^{(0|2)}(z_2, z_3)}{z_1 - z_3} \end{aligned} \quad (2.36)$$

$$\begin{aligned} \frac{y(z_1)}{z_1} \rho_C^{(1|2)}(z_1, z_2) &= 2\rho_C^{(0|2)}(z_1, z_2)\rho_C^{(1|1)}(z_1) + \rho_C^{(0|3)}(z_1, z_1, z_2) + \\ &+ \frac{1}{z_1} \partial_{z_2} \frac{z_1 \rho_C^{(1|1)}(z_1) - z_2 \rho_C^{(1|1)}(z_2)}{z_1 - z_2} \end{aligned} \quad (2.37)$$

$$\frac{y(z_1)}{z_1} \rho_C^{(2|1)}(z_1) = \left( \rho_C^{(1|1)}(z_1) \right)^2 + \rho_C^{(1|2)}(z_1, z_1) \quad (2.38)$$

**Non-Gaussian phases.** A generic phase of the complex matrix model is given by first several time variables shifted,  $t_k \rightarrow T_k + t_k$ . Then, for a generic polynomial potential  $W(z) = \sum_{k=0}^{n+1} T_k z^k$  the Virasoro constraints for the complex-matrix model look like

$$W'(z)\rho(z) = \rho^2(z) + f(z) + g^2 \hat{\nabla}(z)\rho(z) + \frac{1}{z} P_z^- [z v'(z)\rho(z)] \quad (2.39)$$

where

$$f(z) = \frac{1}{z} P_z^+ [z W'(z)\rho(z)] \quad (2.40)$$

Higher resolvents can be extracted from the equations

$$\begin{aligned}
 W'(z)\rho_W^{(p|m+1)}(z, z_1, \dots, z_m) - f_C^{(p|m+1)}(z|z_1, \dots, z_m) = \\
 = \sum_q \sum_{m_1+m_2=m} \rho_W^{(q|m_1+1)}(z, z_{i_1}, \dots, z_{i_{m_1}})\rho_W^{(p-q|m_2+1)}(z, z_{j_1}, \dots, z_{j_{m_2}}) + \\
 + \sum_{i=1}^m \frac{1}{z} \frac{\partial}{\partial z_i} \frac{z\rho_W^{(p|m)}(z, z_1, \dots, \check{z}_i, \dots, z_m) - z_1\rho_W^{(p|m)}(z_1, \dots, z_m)}{z - z_i} + \\
 + \hat{\nabla}(z)\rho_W^{(p-1|m+1)}(z, z_1, \dots, z_m). \tag{2.41}
 \end{aligned}$$

where

$$f_C^{(p|m+1)}(z|z_1, \dots, z_m) = \check{R}_C(z)\rho^{(p|m)}(z_1, \dots, z_m) \tag{2.42}$$

with

$$\check{R}_C(z) = \frac{1}{z} P_z^+ [zW'(z)\check{\nabla}(z)] = - \sum_{a=-1}^{n-1} \sum_{b=0}^{n-a-1} (a+b+2)T_{a+b+2}z^a \frac{\partial}{\partial T_b} \tag{2.43}$$

Quadratic equation for simplest resolvent leads to the answer

$$\rho_C^{(0|1)}(z) = \frac{W'(z) - \frac{y_C(z)}{z}}{2} \tag{2.44}$$

where

$$y_C(z)^2 = z \left( W'^2 - 4\check{R}_C(z)F_C^{(0)} \right) \tag{2.45}$$

For the Gaussian complex model  $T_k = \delta_{k,1}$ , so  $W(z) = z$ ,  $\check{R}_C(z) = -\frac{1}{z} \frac{\partial}{\partial T_0}$

### 3 Kontsevich model $Z_K(\tau)$

#### 3.1 Original integral representation

Kontsevich model was originally defined in [22] as a generating function of topological indices of the moduli space of Riemann surfaces. M.Kontsevich represented this generating function in form of the now-famous matrix integral over auxiliary  $n \times n$  dimensional Hermitian matrices (one can easily introduce into this integral the parameter  $g$  similarly to (1.17), [1] however, for the sake of simplicity, we put here  $g = 1$ ):

$$Z_K = \exp\left(-\frac{2}{3}\text{tr } L^3\right) \frac{\int_{n \times n} dh \exp\left(-\frac{1}{3}\text{tr } h^3 + \text{tr } L^2 h\right)}{\int_{n \times n} dh \exp\left(-\text{tr } L h^2\right)} \tag{3.1}$$

where time-variables are Miwa-transformed:

$$\tau_k = \frac{1}{k} \text{tr } L^{-k} \tag{3.2}$$

If expressed through the  $\tau$ -variables,  $Z_K(\tau)$  is actually independent of auxiliary parameter  $n$ . This model can be further generalized to Generalized Kontsevich Model (GKM) [23],

$$Z_{\text{GKM}} = \frac{\int_{n \times n} e^{-U(L,h)} dh}{\int_{n \times n} e^{-U_2(L,h)} dh} \tag{3.3}$$

where

$$U(L, h) \equiv \text{tr} [\mathcal{V}(L + h) - \mathcal{V}(L) - \mathcal{V}'(L)h] \tag{3.4}$$

and

$$U_2(L, h) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} U(L, \epsilon h) \tag{3.5}$$

is an  $h^2$ -term in  $U$ .  $W$  is here an arbitrary (power series) potential and  $Z_{\text{GKM}}$  is a function of the same ( $W$ -independent) Miwa transform (3.2). Many properties of GKM are in fact independent of the choice of  $\mathcal{V}(h)$ .

In fact, one can also consider the matrix model (3.3) with a different normalization as a function of time variables

$$\mathcal{T}_k = \frac{1}{k} \text{tr} \Lambda^k \tag{3.6}$$

where the matrix  $\Lambda = \mathcal{V}'(L)$  enters in *positive* powers. This function is called the character phase and is considered in detail in [20]. In this paper we restrict ourselves with the Kontsevich phase only, where the time variables are given by (3.2).

Note also that it is often convenient to fix  $\mathcal{V}(h)$  to be a polynomial of  $h$  (polynomial Kontsevich model, [23]) or that of  $h^{-1}$  (antipolynomial Kontsevich model [20]). In this section we consider only the polynomial case, leaving the antipolynomial one until the next section (where it emerges within the context of the unitary matrix model).

### 3.2 Eigenvalue representation

Shifting the integration variable  $h \rightarrow h - L$  one obtains that

$$Z_{\text{GKM}} \sim \int_{n \times n} dh \exp(-\text{tr} \mathcal{V}(h) - \text{tr} \mathcal{V}'(L)h) \tag{3.7}$$

Now using the Itzykson-Zuber formula and (1.3), one can perform integration over the angular variables in this integral:

$$\int_{n \times n} dh \exp(-\text{tr} \mathcal{V}(h) - \text{tr} \mathcal{V}'(L)h) \sim \frac{\det_{ij} F_i(\lambda_j)}{\Delta(\lambda)} \tag{3.8}$$

where  $\lambda_i$  are the eigenvalues of the matrix  $\mathcal{V}'(L)$  and

$$F_i(\lambda) = \int dx x^{i-1} \exp(-\mathcal{V}(x) + \lambda x) \tag{3.9}$$

### 3.3 Virasoro constraints

Straightforward Ward identities for  $Z_K$  are as previously associated with the shift  $h \rightarrow h + \epsilon$  of integration variable  $h$ :

$$\left[ \left( \frac{\partial}{\partial L_{\text{tr}}^2} \right)^2 - L^2 \right] \int_{n \times n} dh \exp \left( -\frac{1}{3} \text{tr} h^3 + \text{tr} L^2 h \right) = 0 \tag{3.10}$$

Now one should take into account the normalization factor and come to the  $\tau$ -variables. Conversion to the  $\tau$ -variables is highly non-trivial, it was first performed in [18] and leads



to the celebrated result [13, 14]:

$$\hat{\mathcal{L}}_n Z_K = 0, \quad \hat{\mathcal{L}}_n = \frac{1}{2} \sum_{\substack{k \geq \delta_{n+1,0} \\ k \text{ odd}}} k \tau_k \frac{\partial}{\partial T_{k+2n}} + \frac{1}{4} \sum_{\substack{a+b=2n \\ a,b \geq 0; a,b \text{ odd}}} \frac{\partial^2}{\partial \tau_a \partial \tau_b} + \\ + \delta_{n+1,0} \cdot \frac{\tau_1^2}{4} + \delta_{n,0} \cdot \frac{1}{16} - \frac{\partial}{\partial \tau_{2n+3}}$$

This proved equivalence of Witten’s topological  $2d$  gravity [12] to  $Z_K$  and — since  $Z_K$  is trivially a KP tau-function — proved that partition function of  $2d$  gravity is indeed a tau-function (as anticipated in [55]). Analogous conversion to  $T$ -variables of the Ward identities for  $Z_{\text{GKM}}$  is even more sophisticated and give rise to  $W$ -constraints (or  $\tilde{W}$ -constraints in the character phase) [19, 20].

### 3.4 Determinant representation and integrability

Now one can take into account all the normalization factors and further transform this determinant (after quite tedious calculation, [23]) to

$$Z_{\text{GKM}} = \frac{\det_{ij} \phi_i(L_j)}{\Delta(L)} \tag{3.11}$$

where

$$\phi_i(L) = \sqrt{\mathcal{V}''(L)} e^{\mathcal{V}(L) - L\mathcal{V}'(L)} F_i(\mathcal{V}'(L)) \tag{3.12}$$

Formula (3.11) if true for any number of Miwa variables (size of the determinant) fixes a KP hierarchy  $\tau$ -function [10, 24] that depends on times

$$\tau_k = \frac{1}{k} \sum_i L_i^{-k} \tag{3.13}$$

provided the asymptotics of  $\phi_i(L)$  at large  $L$  is

$$\phi_i(L) \stackrel{L \rightarrow \infty}{\sim} L^{i-1} (1 + O(1/L)) \tag{3.14}$$

In particular, (3.14) guarantees that  $Z_{\text{GKM}}$  is a function of variables  $\tau_k$  (3.13) and does not depend on their number.

Now if one takes the monomial potential  $\mathcal{V}(h) = h^{p+1}$  (the case of a polynomial potential of degree  $p + 1$  describes a hierarchy equivalent to the monomial case, see details in [56]), the partition function  $Z_{\text{GKM}}$  is a  $\tau$ -function of the  $p$ -reduced KdV hierarchy, which does not depend on times  $\tau_{pk}$  for all  $k$ . In particular, the Kontsevich partition function (3.1) is a KdV  $\tau$ -function depending only on odd times  $\tau_{2k+1}$ . A concrete solution of the KdV hierarchy is fixed by the Virasoro constraints (3.11) (in fact, it is enough to use only the lowest constraint in addition to the KdV hierarchy equations in order to fix the partition function unambiguously).

Note that one could starts from the Virasoro constraints (3.11) instead of the matrix integral. Then, there are much more solutions, the KdV one corresponding only to distinguished solutions of the Dijkgraaf-Vafa type [26, 2nd paper].

Note that one can easily continue the (Generalized) Kontsevich matrix integral to the whole Toda lattice hierarchy adding to  $U(L, h)$  (but not to  $U_2(L, h)$ ) the term

$$\Delta U(L, h) = \Delta \mathcal{V}(L + h) - \Delta \mathcal{V}(L), \quad \Delta \mathcal{V}(h) = \aleph \log h - \sum_k \bar{\tau}_k h^{-k} \quad (3.15)$$

Here  $\aleph$  is the zeroth (discrete) Toda time and  $\bar{\tau}_k$  are the negative Toda times. In the special case of quadratic potential  $\mathcal{V}(h) = h^2$  this matrix integral reduces to the Toda chain, as we observed in s.2.

### 3.5 Kontsevich-Penner representation

Of course, Kontsevich model is already in the Kontsevich form. No parameter  $N$  is obligatory present and no Penner term is needed (until one wants to deal with the (Generalized) Kontsevich integral as with the Toda lattice hierarchy).

### 3.6 Genus expansion and the first multiresolvents

**Generic phase.** Similarly to the Hermitian and complex matrix models, a generic phase of the Kontsevich model is given by first several time variables shifted  $\tau_{2k+1} \rightarrow \tau_{2k+1} + T_k$ . In this case,

$$\hat{\nabla}(z) = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \frac{\partial}{\partial \tau_{2k+1}} \quad (3.16)$$

$$W'(z) = \sum_{k=0}^{n+1} \left(k + \frac{1}{2}\right) T_k z^k \quad (3.17)$$

$$v'(z) = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \tau_{2k+1} z^k \quad (3.18)$$

$$W'(z)\rho(z) = \rho^2(z) + f_K(z) + g^2 \hat{\nabla}(z)\rho(z) + z P_z^- \left[ \frac{v'(z)\rho(z)}{z} \right] + \frac{g^2}{16z} + \frac{(\tau_0 - T_0)^2}{16} \quad (3.19)$$

$$f_K(z) = z P_z^+ \left[ \frac{W'(z)\rho(z)}{z} \right] = g^2 \hat{R}_K(z) \log Z \quad (3.20)$$

$$\hat{R}_K(z) = - \sum_{k=2}^{n+1} \sum_{m=0}^{k-2} \left(k + \frac{1}{2}\right) T_k z^{k-m-1} \frac{\partial}{\partial T_m} \quad (3.21)$$

$$\rho(z) = g^2 \hat{\nabla}(z) \log Z \quad (3.22)$$

$$\begin{aligned} & W'(z)\rho_W^{(p|m+1)}(z, z_1, \dots, z_m) - f_W^{(p|m+1)}(z|z_1, \dots, z_m) = \\ & = \sum_q \sum_{m_1+m_2=m} \rho_W^{(q|m_1+1)}(z, z_{i_1}, \dots, z_{i_{m_1}}) \rho_W^{(p-q|m_2+1)}(z, z_{j_1}, \dots, z_{j_{m_2}}) + \\ & + \sum_{i=1}^m \left( \frac{\partial}{\partial z_i} - \frac{1}{2z_i} \right) \frac{z_i \rho_W^{(p|m)}(z, z_1, \dots, z_i, \dots, z_m) - z \rho_W^{(p|m)}(z_1, \dots, z_m)}{z - z_i} \\ & + \hat{\nabla}(z)\rho_W^{(p-1|m+1)}(z, z_1, \dots, z_m) + \\ & + \frac{\delta_{p,1}\delta_{m,0}}{16z} + \delta_{p,0} \left( \frac{\delta_{m,0}T_1^2}{16} - \frac{\delta_{m,1}T_1}{8z_1} + \frac{\delta_{m,2}}{8z_1z_2} \right) \end{aligned} \quad (3.23)$$

**Gaussian case.** In this case,  $W'(z) = z + \frac{T_0}{2}$ ,  $f^{(k|m)} = 0$

$$\rho^{(0|1)}(z) = \frac{z + \frac{T_0}{2} - \sqrt{z^2 + T_0 z}}{2} \quad (3.24)$$

$$\rho^{(0|2)}(z_1, z_2) = \frac{1}{4(z_1 - z_2)^2} \left( \frac{(z_1 + z_2 + 2T_0)z_1 z_2}{y(z_1)y(z_2)} - (z_1 + z_2) \right) \quad (3.25)$$

$$y(z_1)\rho^{(0|2)}(z_1, z_2) = \left( \frac{\partial}{\partial z_2} - \frac{1}{2z_2} \right) \frac{z_2 \rho^{(0|1)}(z_1) - z_1 \rho^{(0|1)}(z_2)}{z_1 - z_2} - \frac{T_0}{8z_2} \quad (3.26)$$

$$\rho^{(0|3)}(z_1, z_2, z_3) = \frac{z_1^2 z_2^2 z_3^2}{y(z_1)^3 y(z_2)^3 y(z_3)^3} \quad (3.27)$$

$$\begin{aligned} y(z_1)\rho^{(0|3)}(z_1, z_2, z_3) &= \left( \frac{\partial}{\partial z_2} - \frac{1}{2z_2} \right) \frac{z_2 \rho^{(0|2)}(z_1, z_3) - z_1 \rho^{(0|2)}(z_2, z_3)}{z_1 - z_2} + \\ &+ \left( \frac{\partial}{\partial z_3} - \frac{1}{2z_3} \right) \frac{z_3 \rho^{(0|2)}(z_1, z_2) - z_1 \rho^{(0|2)}(z_2, z_3)}{z_1 - z_3} + \frac{1}{8z_1 z_2} \end{aligned} \quad (3.28)$$

$$y(z)\rho^{(1|1)}(z) = \rho^{(0|2)}(z, z) + \frac{1}{16z} \quad (3.29)$$

$$\rho^{(1|1)}(z) = \frac{z^3}{16y(z)^5} \quad (3.30)$$

$$\begin{aligned} y(z_1)\rho^{(1|2)}(z_1, z_2) &= 2\rho^{(0|2)}(z_1, z_2)\rho^{(1|1)}(z_1) + \rho^{(0|3)}(z_1, z_1, z_2) \\ &+ \left( \frac{\partial}{\partial z_2} - \frac{1}{2z_2} \right) \frac{z_2 \rho^{(1|1)}(z_1) - z_1 \rho^{(1|1)}(z_2)}{z_1 - z_2} \end{aligned} \quad (3.31)$$

$$y(z_1)\rho^{(2|1)}(z) = \left( \rho^{(1|1)}(z) \right)^2 + \rho^{(1|2)}(z, z) \quad (3.32)$$

$$\rho^{(2|1)}(z_1, z_2) = \frac{z_1^4 z_2^4 5(z_1 + T_0)^2 + 3(z_1 + T_0)(z_2 + T_0) + 5(z_2 + T_0)^2}{32 y(z_1)^7 y(z_2)^7} \quad (3.33)$$

$$\rho^{(2|1)}(z) = \frac{105}{254} \frac{z^6}{y(z)^{11}} \quad (3.34)$$

## 4 BGW model $\tilde{Z}_K(\tau) = Z_{\text{BGW}}(\tau)$

### 4.1 Original integral representation

Brezin-Gross-Witten (BGW) model is defined as a generating function for all correlators of unitary matrices with Haar measure  $[dU]$ :

$$Z_{\text{BGW}} = \int_{N \times N} [dU] \exp \left( \text{Tr } J^\dagger U + \text{Tr } JU^\dagger \right) \quad (4.1)$$

The integral actually depends only on eigenvalues of Hermitian matrix  $M = JJ^\dagger$ , i.e. on the time-variables of the form  $\tau_k = \text{Tr } (JJ^\dagger)^k$ .

Haar measure  $[dU]$  for unitary matrices is non-linear, it can be reduced to a flat measures in different ways. One possibility is to express  $U$  through Hermitian matrices,  $U = \frac{1+iH}{1-iH}$  [57], which defines  $[dU]$  as the flat Hermitian measure  $dH = \prod_{i,j=1}^N dH_{ij}$  with additional Jacobian factor,  $[dU] = \mathcal{J}(H)dH$ ,  $\mathcal{J} = \det(1 + H^2)^{-N}$ . Another possibility [20,

21, 58] is to impose the constraints on the complex matrices:

$$[dU] = \int d^2\Phi \delta(\Phi\Phi^\dagger - I) = \int_{N \times N} dh e^{-\text{Tr } h} \int_{N \times N} d^2\Phi e^{\text{Tr } h\Phi\Phi^\dagger} \quad (4.2)$$

For certain actions the integral over  $d\Phi$  can be explicitly taken and this gives rise to reformulation of original unitary-matrix model.

### 4.2 Eigenvalue representation

Since technically the most simple way to obtain eigenvalue representations is to start with the Kontsevich-Penner representation of the BGW model, we first consider this representation.

### 4.3 Kontsevich-Penner representation

In variance with all other Kontsevich-Penner representations, that of the BGW model connects the two integrals over the two matrices (unitary and Hermitian ones) of the *same* size, [20]:

$$Z_{\text{BGW}} = \int_{N \times N} dh \exp\left(-\text{tr } h^{-1} + \text{tr } Mh - N \text{tr } \log h\right) \quad (4.3)$$

This makes theory of the BGW model somewhat harder and one sometimes embeds it into the universal BGW model [20] with an arbitrary coefficient in front of the logarithmic term. However, in order to make contact with the BGW model (4.1) one ultimately has to put this coefficient equal to  $-N$ .

#### 4.3.1 Proof I: Ward identities

The simplest Ward identity for  $Z_{\text{BGW}}$  has the form

$$\frac{\partial}{\partial J_{\text{tr}}^\dagger} \cdot \frac{\partial}{\partial J_{\text{tr}}} Z_{\text{BGW}}(J, J^\dagger) = I \cdot Z_{\text{BGW}}(J, J^\dagger). \quad (4.4)$$

or, equivalently, [20]

$$\frac{\partial}{\partial M_{\text{tr}}} M \frac{\partial}{\partial M_{\text{tr}}} Z_{\text{BGW}}(M) = I \cdot Z_{\text{BGW}}(M). \quad (4.5)$$

At the same time, integral (4.3) satisfies the equation

$$\left[ \frac{\partial}{\partial M_{\text{tr}}} M \frac{\partial}{\partial M_{\text{tr}}} + (N - \mathcal{N}) \frac{\partial}{\partial M_{\text{tr}}} + \left( \frac{\partial}{\partial M_{\text{tr}}} \right)^2 \mathcal{V}' \left( \frac{\partial}{\partial M_{\text{tr}}} \right) \right] \times \int_{N \times N} dh e^{\text{tr } (Mh - \mathcal{N} \log h + \mathcal{V}(h))} = 0. \quad (4.6)$$

At  $\mathcal{N} = N$  and  $\mathcal{V}'(h) = 1/h$  (4.5) and (4.6) coincide, which establishes (4.3).

**4.3.2 Proof II: Faddeev-Popov trick**

Another simple way to derive the Kontsevich-Penner representation for the BGW model is to use the trick (4.2), [21]. Indeed,

$$\begin{aligned} Z_{\text{BGW}} &= \int dh e^{-\text{Tr } h} \int d^2\Phi \exp\left(\text{tr } h\Phi\Phi^\dagger + \text{Tr } J^\dagger\Phi + \text{tr } J\Phi^\dagger\right) = \\ &= \int_{N \times N} dh \exp\left(-\text{tr } h + \text{tr } M/h - N \text{tr } \log h\right) \stackrel{h \rightarrow 1/h}{=} \\ &\stackrel{h \rightarrow 1/h}{=} \int_{N \times N} dh \exp\left(-\text{tr } h^{-1} + \text{tr } Mh - N \text{tr } \log h\right) \end{aligned}$$

The BGW model has two phases [20]: the *Kontsevich phase*, where partition function is expanded in negative powers of  $M$  and the *character phase* where expansion goes in positive powers of  $M$  (in the case of BGW these phases are nothing but strong and weak coupling solution of [32] respectively). Below we describe them separately.

**4.4 On direct relation between  $Z_C$  and  $Z_{\text{BGW}}$**

In the Kontsevich-Penner form (2.11) the complex matrix model looks somewhat similar to original form (4.1) of the BGW model. From (2.11) one obtains (representing  $\phi = HU^\dagger$  and  $\phi^\dagger = UH$ ):

$$Z_C = \pi^{N^2-n^2} e^{-\text{tr } \eta\eta^\dagger} \int dH e^{-\text{tr } H^2 + 2N \text{tr } \log H} Z_{\text{BGW}}(\eta^\dagger H^2 \eta) \tag{4.7}$$

This tricky formula is the best direct relation known at present. A more transparent relation is still lacking.

**4.5 Character phase**

In this phase the BGW partition function is considered as a function of the variables

$$\mathcal{T}_k = \frac{1}{k} \text{tr } M^k \tag{4.8}$$

and one has to consider the Universal BGW model, i.e. the Kontsevich integral (4.3) with an arbitrary coefficient of the logarithm, which is a free parameter and not the size of the unitary matrix.

**4.5.1 Virasoro constraints**

Performing the change of variables in (4.6) from  $M$  to  $\mathcal{T}_k$ , one can directly obtain the Virasoro constraints satisfied by the BGW partition function  $Z_{\text{BGW}}^+$  in the character phase:

$$\begin{aligned} \hat{L}_m(N, \mathcal{T}) Z_{\text{BGW}}^+ &= \delta_{m,1} Z_{\text{BGW}}^+, \quad m \geq 1 \\ \hat{L}_m(\alpha, \mathcal{T}) &= \alpha \frac{\partial}{\partial \mathcal{T}_m} + \sum_{k \geq 1} k \mathcal{T}_k \frac{\partial}{\partial \mathcal{T}_{k+m}} + \sum_{a+b=m} \frac{\partial^2}{\partial \mathcal{T}_a \partial \mathcal{T}_b} \end{aligned} \tag{4.9}$$

Therefore, the Ward identity (and its solutions) depends on the size of matrix  $N$ . This means that the integral (4.3) is not just a function of variables  $\mathcal{T}$ , but also depends on  $N$ . The way out is to consider the Universal BGW model given by the integral

$$Z_{\text{UBGW}} = \int_{N \times N} dh \exp \left( -\text{tr } h^{-1} + \text{tr } Mh - \mathcal{N} \text{tr } \log h \right) \quad (4.10)$$

Then, the Virasoro constraints become

$$\hat{L}_m(2N - \mathcal{N}, \mathcal{T}) Z_{\text{UBGW}}^+ = \delta_{m,1} Z_{\text{UBGW}}^+, \quad m \geq 1 \quad (4.11)$$

and choosing  $\mathcal{N} = 2N - \aleph$ , one arrives at the partition function  $Z_{\text{BGW}}^+$  that does not depend on  $N$  (but only on the parameter  $\aleph$ ) though the integrand in (4.10) does!

#### 4.5.2 Determinant representation and integrability

One can easily integrate over the angular variables in (4.10) as in the previous section to obtain

$$Z_{\text{UBGW}} = \frac{\det F_i(M_j)}{\Delta(M)} \quad (4.12)$$

where

$$F_i(M) = \int dh h^{i-1} \exp \left( -\frac{1}{h} + MH - \mathcal{N} \log h \right) = 2\pi i \left( 2\sqrt{M} \right)^{\mathcal{N}-i} I_{\mathcal{N}-i} \left( 2\sqrt{M} \right) \quad (4.13)$$

where  $I_k(z)$  are the modified Bessel functions. After some work [20], this formula can be recast to the form (3.11) with the asymptotics (3.14), where  $L = 1/M$  and

$$\phi_i(M) = \frac{\Gamma(2N - \mathcal{N} - 2i + 2)}{2^{2-i} \pi i} \left( \frac{\sqrt{\mu}}{2} \right)^{2N - \mathcal{N} - 1} I_{N - \mathcal{N}/2 - i} \left( \frac{2}{\sqrt{\mu}} \right) \quad (4.14)$$

Again in order to make these functions independent of  $N$ , one has to choose  $\mathcal{N} = 2N - \aleph$ . At the same time, this proves that, under such a choice,  $Z_{\text{UBGW}}^+$  is a  $\tau$ -function of the KP hierarchy.

#### 4.6 Kontsevich phase

In the Kontsevich phase the unitary matrix integral is considered as a function of variables

$$\tau_k = -\frac{1}{k} \text{tr } M^{-k} \quad (4.15)$$

This time the integral (4.3) does not depend on  $N$  provided it is properly normalized:

$$Z_{\text{BGW}}^+ = e^{-2 \text{tr } M^{1/2}} \sqrt{\frac{\det(M^{1/2} \otimes M + M \otimes M^{1/2})}{(\det M)^N}} Z_{\text{BGW}} \quad (4.16)$$

One can check by a direct (quite involved) calculation [20] that  $Z_{\text{BGW}}^+$  depends only on odd times  $\tau_{2k+1}$ .

#### 4.6.1 Virasoro constraints

Using the Ward identity (4.5) one can now make the change to variables (4.15) to obtain the Virasoro constraints satisfied by  $Z_{\text{BGW}}^+$  [20, 33]:

$$\hat{\mathcal{L}}_m Z_{\text{BGW}}^+ = 0, \quad m \geq 0$$

$$\hat{\mathcal{L}}_m = -\frac{1}{2} \sum_{\text{odd } k} k \tau_k \frac{\partial}{\partial \tau_{k-2m}} + \frac{1}{2} \sum_{\substack{\text{odd } a,b \\ a+b=2m}} \frac{\partial^2}{\partial \tau_a \partial \tau_b} + \frac{\partial}{\partial \tau_{2m+1}} + \frac{\delta_{m,0}}{16} \quad (4.17)$$

#### 4.6.2 Determinant representation and integrability

In the Kontsevich phase, performing integration over the angular variables in (4.3) and taking into account the normalization factor, one obtains [20] the determinant representation (3.11) with the asymptotics (3.14), where

$$\phi_{i-N}(M) = 2\sqrt{\pi} e^{-2M} M^{i-N-1/2} I_{i-N-1}(2M) \quad (4.18)$$

This means that the partition function  $Z_{\text{BGW}}^+$  of the unitary matrix model is a  $\tau$ -function of the KP hierarchy. Moreover, as it was already noted, it does not depend on odd times and is, in fact, a  $\tau$ -function of the KdV hierarchy.

#### 4.7 Genus expansion and the first multiresolvents

**Generic phase.** As before, we shift first time variables  $\tau_{2k+1} \rightarrow \tau_{2k+1} + T_k$ ,

$$\hat{\nabla}(z) = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \frac{\partial}{\partial \tau_k} \quad (4.19)$$

$$W'(z) = \sum_{k=0}^{n+1} \left(k + \frac{1}{2}\right) T_k z^k \quad (4.20)$$

$$v'(z) = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \tau_k z^k \quad (4.21)$$

and obtain the loop equations

$$W'(z)\rho(z) = \rho^2(z) + f_{\text{BGW}}(z) + g^2 \hat{\nabla}(z)\rho(z) + P_z^- [v'(z)\rho(z)] + \frac{g^2}{16z} \quad (4.22)$$

$$f_K(z) = P_z^+ [W'(z)\rho(z)] = g^2 \hat{R}_K(z) \log Z \quad (4.23)$$

$$\hat{R}_K(z) = - \sum_{k=1}^{n+1} \sum_{m=0}^{k-1} \left(k + \frac{1}{2}\right) T_k z^{k-m-1} \frac{\partial}{\partial T_m} \quad (4.24)$$

$$\rho(z) = g^2 \hat{\nabla}(z) \log Z \quad (4.25)$$

$$\begin{aligned}
 & W'(z)\rho_W^{(p|m+1)}(z, z_1, \dots, z_m) - f_W^{(p|m+1)}(z|z_1, \dots, z_m) = \\
 & = \sum_q \sum_{m_1+m_2=m} \rho_W^{(q|m_1+1)}(z, z_{i_1}, \dots, z_{i_{m_1}}) \rho_W^{(p-q|m_2+1)}(z, z_{j_1}, \dots, z_{j_{m_2}}) + \\
 & + \sum_{i=1}^m \left( z_i \frac{\partial}{\partial z_i} + \frac{1}{2} \right) \frac{\rho_W^{(p|m)}(z, z_1, \dots, z_i, \dots, z_m) - \rho_W^{(p|m)}(z_1, \dots, z_m)}{z - z_i} + \\
 & + \hat{\nabla}(z) \rho_W^{(p-1|m+1)}(z, z_1, \dots, z_m) + \frac{\delta_{p,1} \delta_{m,0}}{16z}
 \end{aligned} \tag{4.26}$$

**Gaussian phase.** In this case,  $W'(z) = 1$ ,  $f^{(k|m)} = 0$

$$\rho^{(0|1)}(z) = 0 \tag{4.27}$$

$$\rho^{(0|2)}(z_1, z_2) = 0 \tag{4.28}$$

$$\rho^{(0|3)}(z_1, z_2, z_3) = 0 \tag{4.29}$$

$$\rho^{(1|1)}(z) = \frac{1}{16z} \tag{4.30}$$

$$\rho^{(1|2)}(z_1, z_2) = \left( z_2 \frac{\partial}{\partial z_2} + \frac{1}{2} \right) \frac{\rho^{(1|1)}(z_1) - \rho^{(1|1)}(z_2)}{z_1 - z_2} \tag{4.31}$$

$$\rho^{(2|1)}(z) = \left( \rho^{(1|1)}(z) \right)^2 + \rho^{(1|2)}(z, z) \tag{4.32}$$

$$\rho^{(1|2)}(z_1, z_2) = \frac{1}{32z_1 z_2} \tag{4.33}$$

$$\rho^{(2|1)}(z) = \frac{9}{256z^2} \tag{4.34}$$

**Free energy in the Gaussian case.** As a direct corollary of Virasoro constraints (4.17), one can calculate the free energy expansion in the parameter  $g$ ,  $\log Z_{\text{BGW}} = \sum_{k=0} g^{2k-2} \mathcal{F}_{\text{BGW}}^{(k)}$ :

$$\begin{aligned}
 \mathcal{F}_{\text{BGW}}^{(0)} &= 0 \\
 \mathcal{F}_{\text{BGW}}^{(1)} &= -1/8 \ln(\tau_0 - 2) \\
 \mathcal{F}_{\text{BGW}}^{(2)} &= -\frac{9}{32} \frac{\tau_1}{(\tau_0 - 2)^3} \\
 \mathcal{F}_{\text{BGW}}^{(3)} &= -\frac{225}{64} \frac{\tau_2}{(\tau_0 - 2)^5} + \frac{567}{64} \frac{\tau_1^2}{(\tau_0 - 2)^6} \\
 \mathcal{F}_{\text{BGW}}^{(4)} &= -\frac{55125}{512} \frac{\tau_3}{(\tau_0 - 2)^7} + \frac{388125}{512} \frac{\tau_2 \tau_1}{(\tau_0 - 2)^8} - \frac{64989}{64} \frac{\tau_1^3}{(\tau_0 - 2)^9}
 \end{aligned} \tag{4.35}$$

In general  $\mathcal{F}_{\text{BGW}}^{(p)}$  is a *polynomial*

$$\mathcal{F}_{\text{BGW}}^{(p)} = \sum_{k_1 + \dots + k_m = p-1} c_{k_1, \dots, k_m} Q_{k_1} \dots Q_{k_m} \tag{4.36}$$

of the variables  $Q_k = \frac{\tau_k}{(\tau_0 - 2)^{2k+1}}$ . This is the best illustration of drastic simplicity of the BGW partition function as compared to the Kontsevich and Hermitian cases, where all



$\mathcal{F}^{(p)}$  are sophisticated transcendental functions, and are simplified only in terms of *moment* variables. One may say in the BGW case the moment variables are extremely simple.

## Part II

# Decomposition formulas

### 5 The idea of decomposition formulas [1]

The key observation is that multiresolvents – if defined according to the rule (10) – are polydifferentials on the bare spectral curve  $\Sigma$ , intimately related to the  $\widehat{U(1)}$  current  $\hat{\mathcal{J}}(z)$  on  $\Sigma$ , with prescribed singularities: usually they are allowed at some fixed points (*punctures*) on  $\Sigma$ . In this approach the Virasoro constraints on partition function are written as

$$\hat{\mathcal{P}}_- \left( \hat{\mathcal{J}}^2(z) \right) Z = \oint_C \mathcal{K}(z, z') \left( \hat{\mathcal{J}}^2(z) \right) Z = 0 \tag{5.1}$$

with a certain kernel  $\mathcal{K}(z, z')$ , made out of the free-field Green function on  $\Sigma$ . The current is also "shifted":  $\hat{\mathcal{J}}(z) \longrightarrow \hat{\mathcal{J}}(z) + \Delta\hat{\mathcal{J}}(z)$  and partition function  $Z$  depends on the choice of:

- the complex curve (Riemann surface)  $\Sigma$ ,
- the Green function  $\mathcal{K}(z, z')$ , i.e. projection operator  $\hat{\mathcal{P}}_-$ ,
- the punctures on  $\Sigma$  and associated loop operator  $\hat{\mathcal{J}}(z)$ ,
- the local coordinates in the vicinity of the punctures,
- the involution of the curve with punctures and loop operator,
- the shift  $\Delta\mathcal{J}(z)$  on  $\Sigma$ ,
- the contour  $C$  which separates two sets of punctures.

If contour  $C$  goes around an isolated puncture,  $Z$  is actually defined by its infinitesimal vicinity and depends on behavior (the type of singularity) of  $\hat{\mathcal{J}}(z)$  at this particular puncture. Coordinate dependence is reduced to the action of a unitary operator (Bogoliubov transform, and exponential of bilinear function of  $\hat{\mathcal{J}}$ ) on  $Z$ . Types of singularities and associated  $Z$ 's can be classified, and our quartet  $Z_H, Z_C, Z_K$  and  $Z_{BGW}$  are the lowest members of this classification. The former two are associated with a puncture at regular point of  $\Sigma$ , while the latter two – with that at a second-order ramification point.  $Z_C$  and  $Z_{BGW}$  differ from  $Z_H$  and  $Z_K$  by the choice of projection operator  $\hat{\mathcal{P}}_-$ , i.e. the kernel  $\mathcal{K}(z, z')$ .

If contour  $C$  is moved away from the vicinity of the puncture, it can be decomposed into contours encircling all other punctures: this provides relations between  $Z$ 's of different types, associated with different punctures. If  $\Sigma$  has handles or boundaries, there will be additional contributions, associated with non-contractible contours – the corresponding elementary partition functions are not yet identified and investigated — this seems to be a very interesting problem of its own.

In what follows we present the two simplest examples of this procedure, both associated with  $\Sigma$ , represented as a double-covering of the Riemann sphere with two ramification points. Such  $\Sigma$  is of course also a Riemann sphere, however, representing it as a double-covering provides a simple description of behavior, which we allow  $\hat{\mathcal{J}}(z)$  to have at the two ramification points. The other pair of punctures are chosen at preimages of a regular points ( $z = \infty_{\pm}$  in what follows). After that, depending on the choice of projection operator  $\mathcal{P}_-$  we obtain either a relation between  $Z_H$  and the two Kontsevich models,  $Z_H = \hat{U}_{KK}(Z_K \otimes Z_K)$ , or between  $Z_C$  and the pair: Kontsevich model and BGW model,  $Z_C = \hat{U}_{KBGW}(Z_K \otimes Z_{BGW})$ . These both examples were already described in [1], but here we provide a more targeted and, hopefully, more clear presentation of the subject. Some mistakes of original version are also corrected, in the case of discrepancies from [1] the present version should be trusted.

## 6 The basic currents, shifts and projection operators

These are the data, defining the standard Virasoro constraints (3) and (5) and thus the four models, discussed in the section I above. All the four are defined in vicinity of a particular puncture and do not depend on the global properties of the bare spectral curve  $\Sigma$ .

### 6.1 Hermitian current

This one is used in the definition of (4) and thus of partition functions  $Z_H$  and  $Z_C$ .

$$\hat{J}_H(z|g^2) = d\hat{\Omega}_H(z) = \sum_{k=0}^{\infty} \left( \frac{k}{2} t_k z^{k-1} dz + g^2 \frac{dz}{z^{k+1}} \frac{\partial}{\partial t_k} \right) \tag{6.1}$$

With this current one can immediately associate a bi-differential  $f_J(z, z') = \hat{J}(z)\hat{J}(z') - : \hat{J}(z)\hat{J}(z') :$  where the normal ordering means all  $t_k$  placed to the left of all  $t$ -derivatives. It is related to the central extension  $\widehat{U}(1)$  and is equal to

$$f_H(z, z'|g^2) = g^2 \frac{dzdz'}{2(z-z')^2} \tag{6.2}$$

This bi-differential will play an important role in comparison of global and local currents and, therefore, in construction of conjugation operators in the next subsections 7 and 8.

The further difference between various partition functions comes from different choices of the shift functions  $W(z)$  [26] and projector operators [1],

$$P_m \left[ \sum_{k=-\infty}^{\infty} \frac{a_k dz^2}{z^{k+2}} \right] = \sum_{k=m}^{\infty} \frac{a_k dz^2}{z^{k+2}} \tag{6.3}$$

The two correlated choices lead to the two simplest models, associated with (6.1): to  $Z_H$  and  $Z_C$ .

## 1. Gaussian Hermitian model [26]

This model corresponds to the shift

$$\Delta \hat{J}_H(z) = -\frac{zdz}{2} \quad (6.4)$$

Partition function is completely fixed by Virasoro constraints

$$\hat{T}_H(z)Z_H = 0 \quad (6.5)$$

where

$$\begin{aligned} \hat{T}_H(z) &= P_{-1} \left[ : (\hat{J}_H(z) + \Delta \hat{J}_H(z))^2 : \right] = g^2 \sum_{n=-1} \frac{(dz)^2}{z^{n+2}} \hat{L}_n, \\ \hat{L}_n &= \sum_{k=1}^{\infty} k \left( t_k - \frac{\delta_{k,2}}{2} \right) \frac{\partial}{\partial t_{k+n}} + g^2 \sum_{k=0}^n \frac{\partial^2}{\partial t_k \partial t_{n-k}} \end{aligned}$$

with

$$\frac{\partial}{\partial t_0} Z_H(t) = \frac{S}{g^2} Z_H(t) \quad (6.6)$$

Given (6.4), the choice of  $P_{-1}$  from all the  $P_m$  is distinguished: with this choice only partition function is *unambiguously* defined by (6.5). There are interesting situations, when the choice of  $P_m$  is not adjusted to the shift in this way: the best known example is provided by Dijkgraaf-Vafa partition functions [50, 51], where projector is the same  $P_{-1}$  as in Gaussian model, but the shift  $\Delta \hat{J}_H(z) = dW(z)$  is generated by polynomial  $W(z)$  of degree higher than two.

## 2. Gaussian complex model [30]

This model corresponds to the shift

$$\Delta \hat{J}_C(z) = -\frac{dz}{2} \quad (6.7)$$

Thus partition function is completely fixed by Virasoro constraints

$$\hat{T}_C(z)Z_C = 0 \quad (6.8)$$

where projector is taken to be  $P_0$  — again, to guarantee the uniqueness of the solution to (6.8), — and

$$\begin{aligned} \hat{T}_C(z) &= P_0 \left[ : (\hat{J}_H(z) + \Delta \hat{J}_C(z))^2 : \right] = g^2 \sum_{n=0} \frac{(dz)^2}{z^{n+2}} \hat{L}_n, \\ \hat{L}_n &= \sum_{k=1}^{\infty} k (t_k - \delta_{k,1}) \frac{\partial}{\partial t_{k+n}} + g^2 \sum_{k=0}^n \frac{\partial^2}{\partial t_k \partial t_{n-k}} \end{aligned}$$

with

$$\frac{\partial}{\partial t_0} Z_C(t) = \frac{S}{g^2} Z_C(t) \quad (6.9)$$

## 6.2 Kontsevich current

This one is used in the definition of (6) and thus of partition functions  $Z_K$  and  $Z_{\text{BGW}}$ ,

$$\hat{J}_K(\xi|g^2) = d\hat{\Omega}_K(\xi) = \sum_{k=0}^{\infty} \frac{1}{2} \left(k + \frac{1}{2}\right) \tau_k \xi^{2k} d\xi + g^2 \frac{d\xi}{\xi^{2k+2}} \frac{\partial}{\partial \tau_k} \quad (6.10)$$

is — up to traditional but unimportant change of time-variables — the even part of the current (6.1). However, associated central term bi-differential looks more sophisticated (being the symmetric part of the bi-differential):

$$f_K(\xi, \xi'|g^2) = g^2 \frac{(\xi^2 + \xi'^2) d\xi d\xi'}{4(\xi^2 - \xi'^2)^2} \quad (6.11)$$

The simplest partition functions, associated with this current, are Kontsevich  $\tau$ -function and BGW model.

### 1. Kontsevich $\tau$ -function [22, 23]

This time the shift is

$$\Delta \hat{J}_K = -\frac{\xi^2 d\xi}{2} \quad (6.12)$$

and the relevant projector is  $P_{-2}$ :

$$\begin{aligned} \hat{T}_K(\xi) &= P_{-2} \left[ : (\hat{J}_K + \Delta \hat{J}_K)^2 : \right] = g^2 \sum_{n=-1}^{\infty} \frac{(d\xi)^2}{\xi^{2n+2}} \hat{\mathcal{L}}_n, \\ \hat{\mathcal{L}}_n &= \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \left(\tau_k - \frac{2\delta_{k,1}}{3}\right) \frac{\partial}{\partial \tau_{k+n}} + g^2 \sum_{k=0}^{n-1} \frac{\partial^2}{\partial \tau_k \partial \tau_{n-1-k}} + \frac{\delta_{n,0}}{16} + \frac{\delta_{n,-1} \tau_0^2}{16g^2} \end{aligned}$$

Then  $Z_K$  is uniquely defined by

$$\hat{T}_K(\xi) Z_K = 0 \quad (6.13)$$

### 2. Brezin-Gross-Witten model [20]

Now the shift is

$$\Delta J_{\text{BGW}} = -\frac{cd\xi}{4} \quad (6.14)$$

and

$$\begin{aligned} \hat{T}_{\text{BGW}}(\xi) &= P_0 \left[ : (J_K + \Delta J_{\text{BGW}})^2 : \right] = g^2 \sum_{n=0}^{\infty} \frac{(d\xi)^2}{\xi^{2n+2}} \mathcal{L}_n, \\ \mathcal{L}_n &= \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) (\tau_k - 2\delta_{k,0}) \frac{\partial}{\partial \tau_{k+n}} + g^2 \sum_{k=0}^{n-1} \frac{\partial^2}{\partial \tau_k \partial \tau_{n-1-k}} + \frac{\delta_{n,0}}{16} \end{aligned}$$

with projector  $P_0$  unambiguously specify  $Z_{\text{BGW}}$  by

$$\hat{T}_{\text{BGW}}(\xi) Z_{\text{BGW}} = 0 \quad (6.15)$$

## 7 Decomposition relation $Z_H \longrightarrow Z_K \otimes Z_K$

Now we can select a bare spectral curve  $\Sigma$ :

$$y^2 = z^2 - a^2 \quad (7.1)$$

select the punctures: at  $z = \pm a$  and  $z = \infty_{\pm}$ , and select the global current by allowing specific singularities at punctures:

$$\hat{\mathcal{J}}(z|g^2) = \sum_{k=0} \left(k + \frac{1}{2}\right) (A_k + zB_k)y^{2k-1}dz + g^2(C_k + zD_k)\frac{dz}{y^{2k+3}} \quad (7.2)$$

Bi-differential for this current

$$f_{\mathcal{J}} = g^2 \frac{(zz' - a^2)dzdz'}{2(z - z')^2 y(z)y(z')} \quad (7.3)$$

is defined by commutation relations

$$C_k = a^2 \frac{\partial}{\partial A_k} + \frac{k+1}{k + \frac{3}{2}} \frac{\partial}{\partial A_{k+1}}, \quad D_k = \frac{\partial}{\partial B_k}, \quad (7.4)$$

At punctures it is equivalent to the bi-differentials of the basic currents:

$$\begin{aligned} f_{\mathcal{J}}(z, z') &\stackrel{z \rightarrow \infty_{\pm}}{\sim} f_H(z, z') \\ f_{\mathcal{J}}(z, z') &\stackrel{z \rightarrow \pm a}{\sim} 4f_K(\xi_{\pm}, \xi'_{\pm}) \end{aligned}$$

where  $\xi_{\pm}$  are some local coordinates in the vicinity of ramification points  $a$  and  $-a$ , defined respectively by

$$z = a + \sum_{k=1}^{\infty} \alpha_k^+ \xi_+^{2k} \quad (7.5)$$

and

$$z = -a + \sum_{k=1}^{\infty} \alpha_k^- \xi_-^{2k} \quad (7.6)$$

The current (7.2) itself is equivalent to the currents from section 6:

$$\begin{aligned} \hat{\mathcal{J}}(z) &\stackrel{z \rightarrow \infty_{\pm}}{\sim} \hat{J}_H(z) \\ \hat{\mathcal{J}}(z) &\stackrel{z \rightarrow \pm a}{\sim} 2\hat{J}_K(\xi) \end{aligned}$$

Time-variables in parametrization of the global current are related to local time as follows:

$$\begin{aligned} t_k &\sim \frac{2}{k} \oint \frac{\hat{\mathcal{J}}}{z^k} \\ \frac{\partial}{\partial t_k} &\sim \frac{1}{g^2} \oint z^k \hat{\mathcal{J}} \\ \tau_k &\sim \frac{2}{2k+1} \oint \frac{\hat{\mathcal{J}}}{\xi^{2k+1}} \\ \frac{\partial}{\partial \tau_k} &\sim \frac{1}{2g^2} \oint \xi^{2k+1} \hat{\mathcal{J}} \end{aligned}$$

Global current is related to local currents by conjugation operators. Conjugation operator at infinity is

$$\begin{aligned}
 U_H &= \frac{2}{g^4} \oint_{\infty} \oint_{\infty} (f_{\mathcal{J}}(z, z') - f_H(z, z')) \hat{\Omega}_H(z) \hat{\Omega}_H(z') = \\
 &= \frac{1}{2g^2} \oint_{\infty} \oint_{\infty} \rho_H^{(0|2)}(z, z') v(z) v(z')
 \end{aligned}
 \tag{7.7}$$

where  $\rho_H$  is a bi-differential counterpart of the two-point function of Gaussian Hermitian model

$$\rho_H^{(0|2)}(z, z') = \frac{1}{g^2} (f_{\mathcal{J}}(z, z') - f_H(z, z')) = \frac{1}{2(z_1 - z_2)^2} \left( \frac{z_1 z_2 - a^2}{y(z_1) y(z_2)} - 1 \right)
 \tag{7.8}$$

At ramification points the conjugation operator is as follows

$$\hat{V}_H = \frac{2}{g^4} \sum_{i,j=\pm} \oint_{a_i} \oint_{a_j} (f_{\mathcal{J}}(z, z') - 4\delta_{ij} f_K(\xi_i, \xi_j)) \hat{\Omega}_K(\xi_i) \hat{\Omega}_K(\xi_j)
 \tag{7.9}$$

To establish required Virasoro constraints one should shift the global current

$$\hat{\mathcal{J}} \rightarrow \hat{\mathcal{J}} - \frac{y(z) dz}{2}
 \tag{7.10}$$

which leads to a shift of the conjugation operators:

$$\begin{aligned}
 U_H &\rightarrow U_H + \frac{2}{g^2} \oint \frac{z - y(z)}{2} \hat{\Omega}_H(z) dz = U + \frac{1}{g^2} \oint \rho_{\text{GH}}^{(0|1)}(z) v(z) dz \\
 \hat{V}_H &\rightarrow \hat{V}_H + \frac{2}{g^2} \left( \oint_{\xi_+=0} \left( \frac{2\xi_+^2 d\xi_+ - y(z) dz}{2} \right) \hat{\Omega}_K(\xi_+) + \oint_{\xi_-=0} \left( \frac{2\xi_-^2 d\xi_- - y(z) dz}{2} \right) \hat{\Omega}_K(\xi_-) \right)
 \end{aligned}
 \tag{7.11}$$

Then the projector

$$\oint_C \frac{1}{(z - z') dz'} : (\hat{\mathcal{J}}(z') + \Delta \hat{\mathcal{J}}(z'))^2 :
 \tag{7.12}$$

with contour  $C$  encircles the segment ramification points  $\pm a$  on the spectral curve (but not the point  $z$ ! so that always  $|z| > |z'|$ ) do the job: since

$$\frac{1}{(z - z') dz'} = \sum_{k \geq 0} \frac{(z')^k}{z^{k+1} dz'}
 \tag{7.13}$$

it picks up the terms with  $n \geq -1$  in infinity and since

$$\frac{1}{(z - z') dz'} = \sum_{k \geq 0} \frac{(\xi')^{2k-1}}{2\xi^{2k+2} d\xi'}
 \tag{7.14}$$

it picks up the terms with  $n \geq -1$  for ramification points.

After all we get the decomposition formula<sup>5</sup>

$$Z_H(t) = e^{U_H} e^{\hat{V}_H} Z_K(\tau_+) Z_K(\tau_-)
 \tag{7.15}$$

This kind of formula was anticipated in [59].

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<sup>5</sup>Actually, as it was already indicated in [1], we get a whole family of such formulas, with infinite set of free parameters given by coefficients  $\alpha_k^{\pm}$  in (7.5), (7.6).

## 8 Decomposition relation $Z_C \longrightarrow Z_K \otimes Z_{BGW}$

This decomposition formula was the topic of s.8 of ref. [1], however, it is described there in a too sketchy and partly misleading form. Thus we provide here a more detailed and careful presentation.<sup>6</sup>

The bare spectral curve is

$$y_C^2 = z(z - 4S) \tag{8.1}$$

and the four punctures are chosen at  $z = 0, 4S, \infty_{\pm}$ . Accordingly on this curve we define the global current

$$\mathcal{J}(z) = \sum_{k=0}^{\infty} \left( k + \frac{1}{2} \right) (A_k + zB_k) y_C^{2k-1} dz + g^2 \frac{dz}{y_C^{2k+3}} (C_k + zD_k) \tag{8.2}$$

with commutation relations

$$C_k = 8S^2 \frac{\partial}{\partial A_k} - 2S \frac{\partial}{\partial B_k} + \frac{k+1}{k+\frac{3}{2}} \frac{\partial}{\partial A_{k+1}}$$

$$D_k = \frac{\partial}{\partial B_k} - 2S \frac{\partial}{\partial A_k}$$

and the global bi-differential

$$f_{\mathcal{J}}(z, z') = g^2 \frac{(zz' - 2S(z+z')) dz dz'}{2(z-z')^2 y_C y'_C} =$$

$$= g^2 \frac{2y_C^2 y'^2 + (zz' - 2S(z+z') + 8S^2)(y_C^2 + y'^2)}{2(y_C^2 - y'^2)^2 y_C y'_C} dz dz' \tag{8.3}$$

At punctures this bi-differential is equivalent to the following canonical bi-differentials from section 6:

$$f_{\mathcal{J}}(z, z') \stackrel{z \rightarrow \infty_{\pm}}{\sim} f_H(z, z')$$

$$f_{\mathcal{J}}(z, z') \stackrel{z \rightarrow 4S}{\sim} 4f_K(z, z')$$

$$f_{\mathcal{J}}(z, z') \stackrel{z \rightarrow 0}{\sim} 4f_K(z, z')$$

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<sup>6</sup>In [1] we considered decomposition formula for the complex model, starting from the same spectral curve (7.1) as for the Gaussian Hermitian matrix model

$$y_H^2 = z^2 - 4S$$

with additional puncture in  $z = 0_{\pm}$ . However, the global current which we introduced was singular at  $0_{\pm}$ . Actually, in notations of [1], the proper global current should be defined on the curve

$$y_c^2 = z^2(z^2 - 4S)$$

It is more natural to consider instead the current on

$$y_C^2 = \xi(\xi - 4S)$$

of which the previous one is a double covering  $z = \sqrt{\xi}$  as we do in the present text.

The current has the following behavior:

$$\begin{aligned} \mathcal{J}(z) &\stackrel{z \rightarrow \infty^\pm}{\sim} J_H(z) \\ \mathcal{J}(z) &\stackrel{z \rightarrow 4S}{\sim} 2J_K(\xi_+) \\ \mathcal{J}(z) &\stackrel{z \rightarrow 0}{\sim} 2J_K(\xi_-) \end{aligned}$$

with  $\xi_+$ ,  $\xi_-$  — some local coordinates in the vicinities of  $4S$ ,  $0$  respectively:

$$\begin{aligned} z &= 4S + \sum_{k=1}^{\infty} \alpha_k^+ \xi_+^{2k} \\ z &= \sum_{k=1}^{\infty} \alpha_k^- \xi_-^{2k} \end{aligned}$$

Time-variables of the local currents are expressed through those of the global one in the same way as in section 7:

$$\begin{aligned} t_k &\sim \frac{2}{k} \oint \frac{\mathcal{J}}{z^k} \\ \frac{\partial}{\partial t_k} &\sim \frac{1}{g^2} \oint z^k \mathcal{J} \\ \tau_k &\sim \frac{2}{2k+1} \oint \frac{\mathcal{J}}{\xi^{2k+1}} \\ \frac{\partial}{\partial \tau_k} &\sim \frac{1}{2g^2} \oint \xi^{2k+1} \mathcal{J} \end{aligned}$$

Global current is related to local currents through conjugation operators. Conjugation operator at infinity

$$\begin{aligned} U_C &= \frac{2}{g^4} \oint_{\infty} \oint_{\infty} (f_{\mathcal{J}}(z, z') - f_H(z, z')) \Omega_H(z) \Omega_H(z') = \\ &= \frac{1}{2g^2} \oint_{\infty} \oint_{\infty} \rho_C^{(0|2)}(z, z') v(z) v(z') \end{aligned} \quad (8.4)$$

where  $\rho_C$  is a bi-differential counterpart of the two-point function of Gaussian Hermitian model

$$\rho_C^{(0|2)}(z, z') = \frac{1}{g^2} (f_{\mathcal{J}}(z, z') - f_H(z, z')) \quad (8.5)$$

At ramification points the conjugation operator looks as follows

$$V_C = \frac{2}{g^4} \sum_{i,j=\pm} \oint_{a_i} \oint_{a_j} (f_{\mathcal{J}}(z, z') - 4\delta_{ij} f_K(\xi_i, \xi_j)) \Omega_K(\xi_i) \Omega_K(\xi_j) \quad (8.6)$$

The shift of the current

$$\mathcal{J} \rightarrow \mathcal{J} - \frac{y_C(z) dz}{2z} \quad (8.7)$$

corresponds to the shift of conjugation operators

$$\begin{aligned} U_H &\rightarrow U_H + \frac{2}{g^2} \oint \frac{z - y(z)}{2z} \Omega_H(z) dz = U + \frac{1}{g^2} \oint \rho_C^{(0|1)}(z) v(z) dz \\ V_H &\rightarrow V_H + \frac{2}{g^2} \left( \oint_{\xi_+=0} \left( \xi_+^2 d\xi_+ - \frac{y(z) dz}{2z} \right) \Omega_K(\xi_+) + \oint_{\xi_-=0} \left( \frac{cd\xi_-}{2} - \frac{y(z) dz}{2z} \right) \Omega_K(\xi_-) \right) \end{aligned} \quad (8.8)$$



The difference from the case of Hermitian model is that now we should get

$$\begin{aligned} L_n, \quad n \geq 0 \quad \text{at } \infty, \\ \mathcal{L}_n, \quad n \geq 0 \quad \text{at } a, \\ \mathcal{L}_n, \quad n \geq -1 \quad \text{at } -a \end{aligned}$$

Thus this time the proper projector is

$$\oint_C \frac{z'}{(z-z')dz'} : (\mathcal{J}(z') + \Delta\mathcal{J}(z'))^2 : \tag{8.9}$$

and we finally obtain the decomposition formula for complex model:

$$Z_C(t) = e^{V_C} e^{U_C} Z_K(\tau_+) \tilde{Z}_K(\tau_-) = e^{V_C} e^{U_C} Z_K(\tau_+) Z_{BGW}(\tau_-) \tag{8.10}$$

### Conclusion

In this paper we demonstrated that decomposition formula  $Z_H \rightarrow Z_K \otimes Z_K$  of partition function for Gaussian Hermitian model into two cubic Kontsevich models has as its closest analogue another decomposition:  $Z_C \rightarrow Z_K \otimes Z_{BGW}$  of the Gaussian complex model into the cubic Kontsevich and Brezin-Gross-Witten models. Thus all the four models are indeed the very close relatives, though this is not quite so obvious from their original matrix-integral representations. This paper is therefore an important outcome and summary of many different approaches, worked out during the years of development of matrix-model theory. It brings us one-step closer to providing a unified look at the whole variety of eigenvalue models and building up the *M-theory of matrix models*, suggested in [2].

Technically it adds to content of [1] an identification of partition function, denoted there by  $\tilde{Z}_K$ , with that of the very important BGW model — the generating function of all unitary-matrix correlators. From technical point of view the road is now open for search of two different generalizations: to Dijkgraaf-Vafa models [50, 51], which are not *fully* specified by the Virasoro constraints alone and rely upon intriguing and under-developed theory of *check-operators* [49], and to more interesting unitary-matrix models with Itzykson-Zuber measures and further to Kazakov-Migdal multi-matrix models [34]–[37], important both for Yang-Mills theory and for the theory of integer partitions. Putting all these very different problems into the same context, moreover, underlined by the well established theory of free fields on Riemann surfaces [60], is a challenging and a promising perspective.

Another, but, perhaps, related, open problem is *direct* derivation of decomposition formula (8.10) from integral representations of all the models, bypassing the Virasoro constraints and *D*-module representations. Note that this kind of problem remains unsolved even for the crucially important decomposition  $Z_H = \hat{U}(Z_K \otimes Z_K)$ , describing the double-scaling continuum limit of Hermitian matrix model.

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